MASA'S AND CERTAIN TYPE I CLOSED FACES OF C^*-ALGEBRAS

Lawrence G. Brown

Dedicated to the memory of George W. Mackey

Abstract. Let A be a separable C^*-algebra and A^{**} its enveloping W^*-algebra. A result of Akemann, Anderson, and Pedersen states that if $\{p_n\}$ is a sequence of mutually orthogonal, minimal projections in A^{**} such that $\sum_k^\infty p_n$ is closed, $\forall k$, then there is a MASA B in A such that each $\varphi_n|B$ is pure and has a unique state extension to A, where φ_n is the pure state of A supported by p_n. We generalize this result in two ways: We prove that B can be required to contain an approximate identity of A, and we show that the countable discrete space which underlies the result cited can be replaced by a general totally disconnected space. We consider two special kinds of type I closed faces, both related to the above, atomic closed faces and closed faces with nearly closed extreme boundary. One specific question is whether an atomic closed face always has an "isolated point". We give a counterexample for this and also show that the answer is yes if the atomic face has nearly closed extreme boundary. We prove a complement to Glimm's theorem on type I C^*-algebras which arises from the theory of type I closed faces. One of our examples is a type I closed face which is isomorphic to a closed face of every non-type I separable C^*-algebra and which is not isomorphic to a closed face of any type I C^*-algebra.

AMS subject classification: 46L05

Key words and phrases. C^*-algebra, closed projection, closed face, MASA, q-continuous, pure state, atomic, type I.

0. Introduction.

This paper was inspired by the paper [5] of C. Akemann, J. Anderson, and G. Pedersen. Much of the terminology used in this section is explained in later sections. To explain the connection with [5], we begin with:

Proposition 0.1. *Let A be a C^*-algebra and (p_n) a sequence of mutually orthogonal, minimal (rank one) projections in A^{**}. Let $p = \sum_1^\infty p_n$, and let φ_n be the pure state supported by p_n. Then either of the following hypotheses implies that p is closed:*

(i) ([5, 2.7(1)⇒(2)]). *There is a strictly positive element e such that each φ_n is definite on e and $\varphi_n(e) \to 0$.*
(ii) ([12, Lemma 3]). *There is a strictly positive element e such that $\sum_1^\infty \varphi_n(e) < \infty$.*

In circumstances similar to 0.1, [5] proves the existence of a MASA B such that each $\varphi_n|_B$ has the unique extension property. The hypotheses require that A be non-unital. It is known (see [6, §4]) that a non-unital C^*-algebra A may have MASA's which do not hereditarily generate A, or equivalently which do not contain an approximate identity of A. If the MASA constructed in [5] does not hereditarily generate A, the situation is intuitively unsatisfactory. (See the first paragraph of [5, §2].)

To investigate strengthening the result of [5], consider $\widetilde{A}$, the result of adjoining an identity to A, and the pure state φ_∞ defined by $\varphi_\infty(\lambda 1_{\widetilde{A}} + a) = \lambda$. The existence of a MASA B of A such that each $\varphi_n|_B$, $1 \le n < \infty$, has the unique extension property and such that B hereditarily generates A is equivalent to the existence of a MASA B_1 of $\widetilde{A}$ such that each $\varphi_n|_{B_1}$, $1 \le n \le \infty$, has the unqiue extension property ($B = B_1 \cap A, B_1 = \widetilde{B}$).

Now the hypotheses of [5] imply that $\sum_{n \in I} p_n$ is closed for every subset I of $\mathbb{N}$. Thus $\{p_n : 1 \le n < \infty\}$ has properties analogous to those of the discrete topological space $\mathbb{N}$. But when p_∞, the support projection of φ_∞, is added to the set, the new set resembles the non-discrete space $\mathbb{N} \cup \{\infty\}$. Thus we seek a generalization of the MASA result of [5] based on a class of topological spaces which includes $\mathbb{N} \cup \{\infty\}$. We accomplish this in Corollary 2.4: Let A be a separable C^*-algebra and X a totally disconnected, second countable, locally compact Hausdorff space. Assume that for each x in X, p_x is a minimal projection in A^{**}, with associated pure state φ_x, such that the p_x's are mutually orthogonal and for each closed (compact) subset S of X, $\sum_{x \in S} p_x$ is the atomic part of a closed (compact) projection, p_S, in A^{**}. Then there is a MASA B of A such that B hereditarily generates A and each $\varphi_x|_B$ has the unique extension property. Moreover, each p_S is in B^{**}.

If X is general, the hypotheses of the above result may seem rather stringent. Partly in order to justify the generality of the result, we attempt to investigate the circumstances in which the hypotheses will be satisfied. A first observation is that every element of $C_0(X)$ (respectively, $C_b(X)$) gives rise to an element of $p_X A^{**} p_X$ which is strongly q-continuous (respectively q-continuous) on p_X. (The concept "q-continuous on p" was defined in [7]. In [11], "strongly q-continuous

on p" was defined and "Tietze extension theorems" for both kinds of relative q-continuity were given.) Thus in Section 3 we give some basic results and examples on the subject of how many relatively q-continuous elements are supported by a given closed projection.

We also focus on a more specific question suggested by the theory of scattered C^*-algebras: Suppose that p is an atomic closed projection in A^{**} and that pA^*p is norm separable. Is there a minimal projection p_0 such that $p_0 \leq p$ and $p - p_0$ is closed? Such a p_0 would give an "isolated point" of the closed face $F(p)$ supported by p. This question is related to the special case of 2.4 where X is countable. Clearly, if we seek to prove that certain conditions imply the hypotheses of 2.4, then we must be able to prove that these conditions imply a positive answer to our isolated point question. Note also that when X in 2.4 is countable, then p_X is atomic, and the words "the atomic part of" can be omitted.

We give a counterexample for the isolated point question in Section 3, but we also give a positive result which has the following hypothesis (nearly closed extreme boundary):

$$\text{(NCEB)} \qquad [P(A) \cap F(p)]^- \subset \{0\} \cup [t,1]P(A) \text{ for some } t \text{ in } (0,1].$$

Here $P(A)$ is the pure state space of A, and (NCEB) holds in particular if the set of extreme points of $F(p)$ is (weak*) closed. Lest (NCEB) seem unnatural or excessively strong, we point out a connection with [5, §4]. Circumstances not covered by 0.1 are actually considered in [5]. Suppose $\{\varphi_n : 1 \leq n < \infty\}$ is a collection of mutually orthogonal pure states such that $\varphi_n \xrightarrow{w^*} 0$ and each equivalence class is finite. With the additional assumption that there is a uniform bound on the size of the equivalence classes, the authors of [5] show in §4 that the needed conditions ([5, 2.7(2)]) are satisfied. Without the uniform boundedness hypothesis, we can show easily that [5, 2.7(2)] is equivalent to (NCEB).

Our positive result, which is in Section 4, is roughly that if p is a closed projection satisfying (NCEB), then equivalence of pure states gives a proper closed map from $[P(A) \cap F(p)]^- \setminus \{0\}$ onto a locally compact Hausdorff space X. If pA^*p is norm separable, then X is countable and hence scattered. In general, X need not even be totally disconnected, of course.

There are some technicalities involving direct integral theory required in order to prove that closed subsets of X give rise to closed projections. This is what leads us to the study in Section 5 of type I closed faces, where the face $F(p)$ is called type I if $pA^{**}p$ is a type I W^*-algebra. Obviously every atomic face is type I, and also $F(p)$ is type I when p (is closed and) satisfies (NCEB), at least if A is separable. Our results on type I closed faces are only rudimentary, and we think the concept is worthy of further study.

Partly because theorems are not always discovered in logical order, our efforts to expand on the results of [5] have led us in several directions. The different parts of this paper, though closely related, do not mesh perfectly. In Section 7 we attempt to exhibit the formal relationships among the previous sections. The earlier sections can in large part be read independently of one another, except that Section 6 is

a continuation of Section 4 relying on Section 5. The promised complement to Glimm's theorem is Proposition 5.11.

A preliminary preprint of this paper was circulated several years ago. Some results overlapping with Section 3 have been independently found by E. Kirchberg (cf. [22, Lemma 2.3]).

1. Preliminaries.

A will always denote a C^*-algebra and A^{**} its enveloping W^*-algebra. For h in A^{**}_{sa} and F a Borel set in $\mathbb{R}$ $E_F(h)$ denotes the spectral projection of h for F. For many of our proofs A must be separable, but we rarely require that A be unital. $S(A)$ is the state space of A, $P(A)$ the pure state space, and $Q(A)$ the quasi-state space (the set of positive functionals of norm at most 1). If p is a projection in A^{**}, $F(p) = \{\varphi \in Q(A) : \varphi(1-p) = 0\}$, the norm closed face of $Q(A)$ supported by p. (Elements of A^* are regarded as functionals on A^{**} without notice.) Topological terminolgy regarding A^* refers to the weak* topology unless the contrary is explicitly indicated. A projection p in A^{**} is called *open* ([1]) if it is the support projection of a hereditary C^*-subalgebra of A and *closed* if $1-p$ is open. Effros proved in [15, Theorem 4.8] (cf. [25, 3.10.7]) that p is closed if and only if $F(p)$ is closed, and if so we follow the usual abuse of notation and call $F(p)$ a "closed face of A". Also, p is called *compact* ([4]) if $F(p) \cap S(A)$ is closed or equivalently if p is closed in $\widetilde{A}^{**}$, where $\widetilde{A}$ is the result of adjoining an identity to A.

The *reduced atomic representation*, π, of A is $\oplus_i \pi_i$, where $\{\pi_i\}$ contains one representative of each unitary equivalence class of irreducible representations. Denote by z_{at} the central projection in A^{**} that supports π. Thus $z_{\text{at}} A^{**} \cong \oplus_i B(H_{\pi_i})$, and $(1-z_{\text{at}})A^{**}$ has no type I factor direct summands. The *atomic part* of an element x of A^{**} is $z_{\text{at}}x$, x is *atomic* if $x = z_{\text{at}}x$, $F(p)$ is *atomic* if p is atomic, etc. Also pure states φ and ψ are called *equivalent* if the irreducible representations π_φ, π_ψ are unitarily equivalent.

We want to comment further on Proposition 0.1. In fact 2.7(1) of [5] actually states that $\sum \varphi_n(e) < \infty$ rather than $\varphi_n(e) \to 0$. However, the proof in [5] that (1) implies (2) uses only the weaker hypothesis, so that it is correct to attribute 0.1(i) to [5]. (Unfortunately, when he was writing [12], the author had not yet read the proofs in [5].) Here is a generalization of 0.1:

Lemma 1.1. *Let (p_n) be a sequence of mutually orthogonal minimal projections in A^{**} and $p = \sum_1^\infty p_n$. If, $\forall a \in A$, $\pi^{**}(p)\pi(a)\pi^{**}(p)$ is a compact operator on H_π, where π is the reduced atomic representation of A, then p is closed.*

The proof of 1.1 and the fact that it implies 0.1(ii) is identical to the proof of Lemma 3 in [12]. Lemma 1.1 implies 0.1(i) because in that case $\pi^{**}(p)\pi(e)\pi^{**}(p)$ is a diagonal operator whose matrix elements approach zero. (If A is σ-unital, it is enough to verify the compactness for a strictly positive element of A, as shown in [12].) Lemma 1.1 also applies under the Standing Assumptions of [5, §4], since then $\pi^{**}(p)\pi(a)\pi^{**}(p)$ is a block-diagonal operator with bounded block size - in particular it is a $(2N+1)$-diagonal operator.

Despite this, we offer the following new proof of 0.1(i), which may be instructive:

The hypothesis that φ_n is definite on e is equivalent to $p_n e = e p_n$. Thus if $\lambda_n = \varphi_n(e)$, then $p_n \le E_{\{\lambda_n\}}(e)$. Let $\epsilon_k = \sup\{\lambda_n : n > k\}$ and $q_k = \sum_1^k p_n \vee E_{[0,\epsilon_k]}(e)$. Since $[0, \epsilon_k]$ is a closed set, $E_{[0,\epsilon_k]}(e)$ is closed, and thus [1, Theorem II.7] implies that q_k is closed. Since $E_{\{0\}}(e) = 0$, $p = \wedge_1^\infty q_k$, and [1, Proposition II.5] implies p is closed.

Theorem 1.2. *Let (p_n) be a sequence of mutually orthogonal minimal projections in A^{**} and $p = \sum_1^\infty p_n$. Then the following are equivalent:*

(i) *Every subprojection of p in A^{**} is closed.*
(ii) *$\sum_{n\in I} p_n$ is closed for each subset I of $\mathbb{N}$.*
(iii) *$\sum_k^\infty p_n$ is closed, $\forall k$* (cf. [5, 2.7(2)]).
(iv) *$\pi^{**}(p)\pi(A)\pi^{**}(p) \subset \mathcal{K}(H_\pi)$, where π is the reduced atomic representation of A.*

Proof. (i)⇒(ii)⇒(iii) is obvious.

(iii)⇒(iv): Let $q_k = \sum_k^\infty p_n$. Then $F(q_k)$ is a closed subset of $Q(A)$ and $\bigcap_1^\infty F(q_k) = \{0\}$. By this and [5, 2.3], (q_k) approaches infinity in the sense of [5]. By definition, $\|aq_k\| \to 0$, $\forall a \in A$. Therefore $\pi(a)\pi^{**}(p - q_k) \to \pi(a)\pi^{**}(p)$ in norm. Since $\pi^{**}(p - q_k)$ is a finite rank operator, this implies $\pi(a)\pi^{**}(p)$, and *a fortiori* $\pi^{**}(p)\pi(a)\pi^{**}(p)$, is compact.

(iv)⇒(i): Assume $p' \in A^{**}$ and $p' \le p$. Then $\pi^{**}(p')\pi(A)\pi^{**}(p') \subset \mathcal{K}(H_\pi)$. Clearly there are mutually orthogonal minimal projections p'_n such that $p' = \sum_1^\infty p'_n$. Thus p' is closed by 1.1.

Perhaps it should also be mentioned that if I in 1.2(ii) is finite, then $\sum_{n\in I} p_n$ is finite rank and hence compact ([1, Corollary II.8]).

If φ is in $P(A)$ and B is a C^*−subalgebra of A, we say that $\varphi|_B$ has the *unique extension property* (UEP) if $\varphi|_B \in P(B)$ and φ is the only element of $S(A)$ which extends $\varphi|_B$. The next proposition is probably not original (see [5, p. 267]).

Proposition 1.3. *Assume p is minimal projection in A^{**} and φ is the associated pure state. If B is a C^*−subalgebra of A, then $\varphi|_B$ has (UEP) if and only if p is in B^{**}.*

Proof. Of course B^{**} is identified with the weak* closure of B in A^{**}. First assume (UEP) and let q be the support projection of $\varphi|_B$, so that q is a minimal projection in B^{**}. If ψ is in $S(A)$ and $\psi(q) = 1$, then $\psi|_B$ is in $F(q) \cap S(B)$, and hence $\psi|_B = \varphi|_B$. By (UEP), $\psi = \varphi$. Thus we have shown that $F(q)$, computed in A^*, is one dimensional, and this clearly implies $q = p$.

Conversely, assume $p \in B^{**}$. Since p is minimal in A^{**}, it is clearly minimal in B^{**}. Since $\varphi|_B(p) = 1$ and $\|\varphi|_B\| \le \|\varphi\| = 1$, $\varphi|_B$ is a state supported by p. Therefore $\varphi|_B \in P(B)$. If $\psi \in S(A)$ and $\psi|_B = \varphi|_B$, then ψ and φ agree also on B^{**}. Thus $\psi(p) = \varphi(p) = 1$, ψ is supported by p, and hence $\psi = \varphi$.

Recall the condition (NCEB), which was defined in Section 0 for any projection p in A^{**}. It is also convenient to have a name for the special case of (NCEB) where

$t = 1$.

(CEB) $$[P(A) \cap F(p)]^- \subset \{0\} \cup P(A).$$

The phrase "closed extreme boundary", is accurate only when p is closed, but the main uses of (NCEB) and (CEB) are for projections known *a priori* to be closed.

Theorem 1.4. *Let (p_n) be a sequence of mutually orthogonal minimal projections in A^{**}, (φ_n) the associated sequence of pure states, and $p = \sum_1^\infty p_n$. If the equivalence classes of $\{\varphi_n\}$ are finite and $\varphi_n \xrightarrow{w^*} 0$, then the following are equivalent:*

(i) $\sum_k^\infty p_n$ *is closed,* $\forall k$.
(ii) p *satisfies (NCEB).*
(iii) p *satisfies (CEB).*
(iv) $[F(p) \cap P(A)]^- \subset \{0\} \cup [t, 1]S(A)$ *for some t in* $(0, 1]$.

Proof. Let $\Gamma_1, \Gamma_2, \dots$ be the equivalence classes of $\{\varphi_n\}$, and let $q_i = \sum_{\varphi_n \in \Gamma_i} p_n$. Thus each q_i is a finite rank, and hence compact, projection in A^{**}.

(iv)$\Rightarrow$ (i): For this it is clearly permissible to simplify the notation by assuming $k = 1$. Thus we need to show that p, which is $\sum_i q_i$, is closed. According to Proposition 4.2 of [5], for this it is sufficient to show that (q_i) approaches infinity. Let U be a convex neighborhood of 0 in A^*. We need to find i_0 such that $F(q_i) \subset U$ for $i \geq i_0$. By the Krein-Milman theorem, it is sufficient to show $F(q_i) \cap P(A) \subset U$ for $i \geq i_0$. If this is false we can find nets (ψ_j) and (i_j) such that $\psi_j \in F(q_{i_j}) \cap P(A)$, $i_j \to \infty$, $\psi_j \to \psi$, and $\psi \neq 0$.

Let π be the reduced atomic representation of A and H_j the range of $\pi^{**}(q_{i_j})$. Thus dim $H_j = |\Gamma_j|$. If dim $H_j = 1$ for arbitrarily large j, then $\psi_j = \varphi_n$ for $\varphi_n \in \Gamma_{i_j}$; and we already know $\varphi_n \to 0$. Thus we may assume dim $H_j \geq 2$, $\forall j$. Then we can find unit vectors u_j, v'_j, v''_j in H_j such that $(v'_j, v''_j) = 0$, the pure states $(\pi(\cdot)v'_j, v'_j)$ and $(\pi(\cdot)v''_j, v''_j)$ are in Γ_{i_j}, and $\psi_j = (\pi(\cdot)u_j, u_j)$. Choose a unit vector v_j in span$\{v'_j, v''_j\}$ such that $(v_j, u_j) = 0$, and let $\theta_j = (\pi(\cdot)v_j, v_j)$. Then $\theta_j \to 0$. This follows from Lemma 4.1 of [5], with the N of [5] being 2, or it can be proved directly using an argument similar to the one below. Let $f_j = (\pi(\cdot)u_j, v_j)$, which is an element of A^*. By the Schwarz inequality, $|f_j(a)| \leq \|\pi(a^*)v_j\| = \theta_j(aa^*)^{1/2}$. Therefore $f_j \to 0$. Then if $w_j = ru_j + sv_j$, with $|r|^2 + |s|^2 = 1$, and $\rho_j = (\pi(\cdot)w_j, w_j)$, we see that $\rho_j \in F(p) \cap P(A)$, and $\rho_j \to |r|^2\psi$. We can choose r, s such that $0 < |r|^2\|\psi\| < t$, in contradiction to (iv).

(i)$\Rightarrow$(iii): Suppose $\psi_j \in F(p) \cap P(A)$ and $\psi_j \to \psi$. Then for each j there is i_j such that $\psi_j \in F(q_{i_j})$. If $i_j \nrightarrow \infty$, then by passing to a subnet we may assume $i_j = i$, $\forall j$. Then it is easy to see that $\psi \in F(q_i) \cap P(A)$. (Each ψ_j is a vector state coming from H_j and the unit sphere of H_j is norm compact.) If $i_j \to \infty$, then for each k, $\psi_j \in F(\sum_k^\infty p_n)$ for sufficiently large j. By (i), $\psi \in F(\sum_k^\infty p_n)$. Since $\bigwedge_{k=1}^{\infty} \sum_k^\infty p_n = 0$, $\psi = 0$.

(iii)$\Rightarrow$(ii)$\Rightarrow$(iv) is obvious.

2. Existence of MASA's

Lemma 2.1. *Let A be a C^*-algebra and $\widetilde{A}$ the result of adjoining a new identity to A (i.e., $\widetilde{A} \cong A \oplus \mathbb{C}$ if A is already unital). Let φ_∞ in $P(\widetilde{A})$ be defined by $\varphi_\infty(\lambda 1_{\widetilde{A}} + a) = \lambda$. Assume B_1 is a unital C^*-subalgebra of $\widetilde{A}$ such that $\varphi_\infty|_{B_1}$ has (UEP), and let $B = B_1 \cap A$. Then B hereditarily generates A and $B^{**} = B_1^{**} \cap A^{**}$.*

Proof. That $B^{**} = B_1^{**} \cap A^{**}$ follows, for example, from general Banach space theory and the fact that B_1/B is finite dimensional. Now let p_∞ be the support projection of φ_∞. Then $p_\infty \in B_1^{**}$ by 1.3. Since $1_{\widetilde{A}} \in B_1^{**}$, $1_{\widetilde{A}} - p_\infty$ is also in B_1^{**}, and of course $1_{\widetilde{A}} - p_\infty$ is the identity of A^{**}. Thus $1_{\widetilde{A}} - p_\infty \in B^{**}$, and this implies that B hereditarily generates A.

Lemma 2.2. *Let A be a separable unital C^*-algebra, p a closed projection in A^{**}, and q an open projection in A^{**} such that $q \geq p$. Let $B = her(q)$, the hereditary C^*-subalgebra of A supported by q, and let U be a neighborhood of $F(p) \cap S(A)$ in $S(A)$. Then there is a closed projection p' in B^{**} such that $p'p = 0$ and $\varphi(p') = 0$ implies $\varphi \in U$ for φ in $S(B)$.*

Proof. As usual, we identify B^{**} with $qA^{**}q$ and $S(B)$ with $\{\varphi \in S(A) : \varphi(q) = 1\}$. (The weak* topologies of A^* and B^* agree on $S(B)$.) By Akemann's Urysohn lemma, [2, Theorem 1.1], there is a in A_{sa} such that $p \leq a \leq q$. Then $a \in B$. Let $C = \mathrm{her}(q - p)$, and let e be a strictly positive element of C. Let $b = a - aea$. Then, by an argument of Akemann [3, 1.1], $E_{\{1\}}(b) = p$. Let $p'_n = E_{(-\infty, 1-n^{-1}]}(b)$, where the spectral projection is computed in B^{**}. We claim that for n sufficiently large the choice $p' = p'_n$ suffices. If not, for each n there is φ_n in $S(B)$ such that $\varphi_n(p'_n) = 0$ and $\varphi_n \notin U$. Then φ_n is supported by $q - p'_n = E_{(1-n^{-1}, 1]}(b) \leq E_{[1-n^{-1}, 1]}(b)$. Let φ be a cluster point of (φ_n) in $S(A)$. Then since each $E_{[1-n^{-1}, 1]}(b)$ is closed in A^{**}, φ is supported by $\bigwedge_{n=1}^{\infty} E_{[1-n^{-1}, 1]}(b) = E_{\{1\}}(b) = p$. Therefore $\varphi \in F(p) \cap S(A)$, a contradiction since $\varphi_n \notin U$.

Theorem 2.3. *Let A be a separable C^*-algebra and X a second countable, totally disconnected, locally compact Hausdorff space. Assume that for each x in X, p_x is an atomic projection in A^{**}, the p_x's are mutually orthogonal, and for every closed (compact) subset S of X there is a closed (compact) projection p_S such that $z_{\mathrm{at}} p_S = \sum_{x \in S} p_x$. Then there is a MASA B in A such that B hereditarily generates A and each p_S is in B^{**}.*

Proof. First we reduce to the case A unital, X compact. To do this, let $\widetilde{A}$ be the result of adjoining a new identity to A, and let $\widetilde{X} = X \cup \{\infty\}$ be the one point compactification. If we let p_∞ be as in 2.1, all hypotheses of the theorem are satisfied for $\widetilde{X}$, $\widetilde{A}$. (If S is a compact subset of X, then p_S is compact in A^{**} and hence closed in $\widetilde{A}^{**}$. Any other closed subset of $\widetilde{X}$ is $S \cup \{\infty\}$ for some closed subset S of X. The fact that p_S is closed in A^{**} implies that $p_S + p_\infty$ is closed in $\widetilde{A}^{**}$. Since $\widetilde{A}$ is unital, "closed" and "compact" mean the same for projections in $\widetilde{A}^{**}$.) If B_1 satisfies the conclusion of the theorem for $\widetilde{A}$, $\widetilde{X}$, then by 2.1, $B_1 \cap A$ satisfies the conclusions of the theorem for A, X. Thus from now on we assume A unital and X compact.

Let C be the usual middle-thirds Cantor set in $[0,1]$. Then there is a one-to-one continuous function $f : X \to C$. We will let α and β denote finite strings of $+$'s and $-$'s, and $|\alpha|$ denote the length of α. Let C_+, C_- be the right and left halves of C, C_{++}, C_{+-} the right and left halves of C_+, etc. Let $p_\alpha = p_{f^{-1}(C_\alpha)}$, and let $F_\alpha = S(A) \cap F(p_\alpha)$. Note that $p_{\alpha+}p_{\alpha-} = 0$ and $p_\alpha = p_{\alpha+} + p_{\alpha-}$. This follows, for example, from the theory of universally measurable elements of A^{**}, [25, 4.3] and the fact that the relations are satisfied by the atomic parts of the projections. Let e be a strictly positive element of $\mathrm{her}(1-p_X)$. We are going to construct recursively b_α in A_+ and an open projection q_α in A^{**} such that:

1. $b_{\alpha+}b_{\alpha-} = 0$
2. $p_\alpha \le q_\alpha \le E_{\{1\}}(b_\alpha)$
3. $b_{\alpha+}, b_{\alpha-} \in \mathrm{her}(q_\alpha)$. (Thus $b_\alpha b_{\alpha\pm} = b_{\alpha\pm}$.)
4. If φ in $S(A)$ is supported by $E_{\{1\}}(b_{\alpha\pm})$, then $\varphi(e) < |\alpha|^{-1}2^{-|\alpha|}$.

Fix non-negative functions g_+, g_- in $C([-1,1])$ such that $g_+ = 1$ on $\left[\frac{2}{3},1\right]$, g_+ is supported on $\left[\frac{1}{3},1\right]$, $g_- = 1$ on $\left[-1,-\frac{2}{3}\right]$, and g_- is supported on $\left[-1,-\frac{1}{3}\right]$.

Step 1, $|\alpha| = 1$. Then 3 and 4 are vacuous. Choose a in A_{sa} such that $-1 \le a \le 1$, $p_- \le E_{\{-1\}}(a)$, and $p_+ \le E_{\{1\}}(a)$. This is easily accomplished by [2, Theorem 1.1] and the continuous functional calculus. Let $b_\pm = g_\pm(a)$, $q_+ = E_{(\frac{2}{3},1]}(a)$, and $q_- = E_{[-1,-\frac{2}{3})}(a)$.

Step k, $|\alpha| = k > 1$. We construct $b_{\beta\pm}$, $q_{\beta\pm}$ for each β with $|\beta| = k-1$, assuming of course that b_β, q_β have already been constructed. Apply 2.2 to find a closed projection p' in $\mathrm{her}(q_\beta)^{**}$ such that $p'p_\beta = 0$ and if φ in $S(A)$ is supported by q_β and $\varphi(p') = 0$, then $\varphi(e) < |\beta|^{-1}2^{-|\beta|}$. Next choose a in $\mathrm{her}(q_\beta)$ such that $-1 \le a \le 1$, $p' \le E_{\{0\}}(a)$, and $p_{\beta\pm} \le E_{\{\pm1\}}(a)$. The existence of a could be deduced from [11, 3.43], but it is more elementary to apply Akemann's Urysohn lemma for $\mathrm{her}(q_\beta)$ twice to obtain a_1 and a_2 with $p_{\beta+} \le a_1 \le 1-(p'+p_{\beta-})$ and $p_{\beta-} \le a_2 \le 1-(p'+p_{\beta+})$. Then let $a = a_1 - a_2$. Then let $q_{\beta+} = E_{(\frac{2}{3},1]}(a)$, $q_{\beta-} = E_{[-1,-\frac{2}{3})}(a)$, and $b_{\beta\pm} = g_\pm(a)$.

Now $\{b_\alpha\}$ is commutative, since for $\alpha \ne \alpha'$ either $b_\alpha b_{\alpha'} = 0$, $b_\alpha b_{\alpha'} = b_{\alpha'}$, or $b_\alpha b_{\alpha'} = b_\alpha$. Let B be any MASA containing all b_α's. If $p'_\alpha = E_{\{1\}}(b_\alpha)$, then $p'_\alpha \in B^{**}$. Note that $p'_{\alpha_1}p'_{\alpha_2} = 0$ if $|\alpha_1| = |\alpha_2|$ and $\alpha_1 \ne \alpha_2$ and that $p'_\alpha \ge p_\alpha$.

We show that $p_X \in B^{**}$ by proving $p_X = \bigwedge_{n=1}^{\infty} \bigvee_{|\alpha|=n} p'_\alpha$. Clearly the latter is at least p_X. Suppose $\varphi \in S(A) \cap F(\bigvee_{|\alpha|=n} p'_\alpha)$. Let $\varphi_\alpha = p'_\alpha\varphi p'_\alpha$. Then $\sum_{|\alpha|=n} \|\varphi_\alpha\| = 1$, $\varphi_\alpha(e) < (n-1)^{-1}2^{-(n-1)}\|\varphi_\alpha\|$, by 4, and $\varphi \le 2^n \sum_{|\alpha|=n} \varphi_\alpha$. Therefore $\varphi(e) < 2(n-1)^{-1}$. If the above is true for all n, then $\varphi(e) = 0$ and hence $\varphi \in F(p_X)$.

Finally, to show that every p_S is in B^{**}, note that every closed subset of C is the intersection of a sequence of clopen sets and every clopen set is the union of finitely many C_α's. Thus it is sufficient to show that each p_α is in B^{**}. We do this by showing that $p_\alpha = p_X \wedge p'_\alpha$. This follows from $p'_\alpha \ge p_\alpha$, $p'_\alpha p_\beta = 0$ if $|\alpha| = |\beta|$ and $\alpha \ne \beta$, and $p_X = \sum_{|\beta|=|\alpha|} p_\beta$.

Corollary 2.4. *Assume the hypotheses of 2.3 and also that each p_x is a minimal projection in A^{**}. Let φ_x be the pure state supported by p_x. Then if B is the MASA of 2.3, $\varphi_x|B$ has the unique extension property, $\forall x \in X$.*

Proof. Combine 2.3 and 1.3, and note that $p_x = p_S$ for $S = \{x\}$.

Remark 2.5. Since the construction of the MASA in 2.3 requires only the p_S's, we could start with a more general, but also more abstract, setup, an assignment $S \mapsto p_S$, for S closed, such that:
(i) $p_{S_1} p_{S_2} = p_{S_2} p_{S_1}$,
(ii) $p_\emptyset = 0$,
(iii) $p_{S_1 \cup S_2} = p_{S_1} \vee p_{S_2}$,
(iv) $p_{\cap_1^\infty S_n} = \bigwedge_1^\infty p_{S_n}$, and
(v) p_S is closed and S compact implies p_S compact.
Because of our assumption that X is totally disconnected, condition (i) is redundant. These conditions do not imply that $z_{\mathrm{at}} p_S = \sum_{x \in S} z_{\mathrm{at}} p_{\{x\}}$, and this last property is not needed to construct the MASA. It was used in the proof of 2.3 to prove conditions (iii) and (iv).

Another alternative formulation, using relative $q-$continuity, appears below in 7.1 (see also 7.5). The hypotheses actually used in 2.3 and 2.4 imply a stronger relationship between the structure of $F(p_X)$ and the space X.

3. Relative q-continuity

Let p be a closed projection in A^{**} and h an element of $pA_{\mathrm{sa}}^{**}p$. Then h is called *q-continuous on p* ([7]) if $E_F(h)$ is closed for every closed subset F of $\mathbb{R}$, where the spectral projection is computed in $pA^{**}p$, and h is called *strongly q-continuous on p* ([11]) if in addition, $E_F(h)$ is compact when F is closed and $0 \notin F$. It was shown in [11, 3.43] that h is strongly q-continuous on p if and only if $h = pa$ for some a in A_{sa} such that $pa = ap$, and if A is σ-unital, then h is q-continuous on p if and only if $h = px$ for some x in $M(A)_{\mathrm{sa}}$ such that $px = xp$.

It was neglected in [11] to give any serious examples or discussion of how extensive is the set of relatively q-continuous elements. For general h in $pA^{**}p$ let us say that h is q-continuous or strongly q-continuous on p if both Re h and Im h are. Let $SQC(p) = \{h \in pA^{**}p : h \text{ is strongly } q\text{-continuous on } p\}$, and let $QC(p) = \{h \in pA^{**}p : h \text{ is } q\text{-continuous on } p\}$. By [11, 3.45], $SQC(p)$ is a C^*-algebra, and if A is σ-unital, $QC(p)$ is also a C^*-algebra. We say that p satisfies $(MSQC)$ (many strongly q-continuous elements) if $SQC(p)$ is σ-weakly dense in $pA^{**}p$ and p satisfies (MQC) if $QC(p)$ is σ-weakly dense in $pA^{**}p$. The von Neumann and Kaplansky density theorems give many equivalent formulations of $(MSQC)$, and also (MQC) if A is σ-unital. As for the other extreme, we always have $\mathbb{C}p \subset QC(p)$ and $0 \in SQC(p)$. We will show that $QC(p)$ and $SQC(p)$ need not be any bigger. Of course, $QC(p) = SQC(p)$ if and only if p is compact.

Theorem 3.1. *If p is a closed projection in A^{**}, then the following are equivalent:*

1. *p satisfies $(MSQC)$.*
2. *$pAp = SQC(p)$.*
3. *pAp is an algebra.*

4. *pAp is a Jordan algebra.*
5. *$F(p)$ is isomorphic to the quasi-state space of a C^*-algebra.*

Remarks. If F_1 and F_2 are closed faces of C^*-algebras, we say they are *isomorphic* if there is a 0-preserving affine isomorphism which is also a (weak*) homeomorphism. An intrinsic characterization of pAp was observed in [11] (a portion of the proof of 3.5 for which no originality was claimed): pAp is the set of continuous affine functionals vanishing at 0 on $F(p)$. With help of [15] one can find intrinsic characterizations of $QC(p)$ and $SQC(p)$. One of the consequences of [7, 4.4, 4.5] is that $pA^{**}p$ is the bidual of the Banach space pAp. In [14] we will give an intrinsic characterization of $pM(A)p$. Thus many questions concerning a closed face of a C^*-algebra A can be treated intrinsically, without knowing what A is.

The C^*-algebra of 5 is determined only up to Jordan $*-$isomorphism.

Proof. $1 \Rightarrow 2$: Since $SQC(p) \subset pAp$ and $pA^{**}p$ is the bidual of pAp, $SQC(p)$ is dense in pAp in the weak Banach space topology. Therefore $SQC(p)$ is norm dense in pAp. But $SQC(p)$ is norm closed (since it is a C^*-algebra, for example).

$2 \Rightarrow 3 \Rightarrow 4$: Obvious.

$4 \Rightarrow 1$: Let $a \in A_{\text{sa}}$. Then $papap \in pAp$. Let $(e_i)_{i \in D}$ be an approximate identity of $\text{her}(1-p)$. Then $pa(1-e_i)ap \to papap$. By Dini's theorem for continuous functions on $F(p)$, this convergence is uniform. Thus $\|pa(1-e_i-p)ap\| \to 0$, $\|(1-e_i-p)^{1/2}ap\| \to 0$, and $\|(1-e_i-p)ap\| \to 0$. It follows that $(1-p)ap \in Ap$, since Ap is closed by an argument similar to [7, 4.4]. If $(1-p)ap = xp$ for x in A, then $pxp = 0$. Therefore $x \in L + R$, where $L = \{b \in A : bp = 0\}$ and $R = L^* = \{b \in A : pb = 0\}$, (proof of [7, 4.4]). Since $Lp = 0$, $(1-p)ap = rp$ for some r in R. Then if $a' = a - r - r^*$, $pa'p = pap$ and $a'p = pa'$. Thus $pap \in SQC(p)$. This shows 2, but since pAp is σ-weakly dense in $pA^{**}p$, it is obvious that 2 implies 1.

That 3 implies 5 is obvious from previous remarks and is also essentially included in the proof of [7, 4.5].

That 5 implies 2 is also obvious from previous remarks and the fact ([4, Theorem III.3]) that 2 is true when $p = 1$.

Theorem 3.2. *Let A be a σ-unital C^*-algebra, p a closed projection in A^{**}, and let $B = SQC(p)$. If B is non-degenerately embedded in $pA^{**}p$, then $M(B)$ is naturally isomorphic to $QC(p)$.*

Remarks. 1. When $B = pAp$ (i.e., when the conditions of 3.1 hold), this result was partly proved in [7, 4.5].

2. It follows from 3.2 that if $SQC(p)$ is non-degenerate in $pA^{**}p$ and if p does not satisfy (MSQC), then p does not satisfy (MQC). This is so because $M(B) \subset B''$.

Proof. Let A^{**} be represented on H via the universal representation of A. The non-degeneracy hypothesis means that B is non-degenerately represented on pH. Therefore $M(B)$ is isomorphic to the idealizer of B in $B(pH)$. It follows that if F is a closed subset of $\mathbb{R}$ and h is in $M(B)_{\text{sa}}$ then there is a hereditary C^*-subalgebra B_0 of B such that any approximate identity of B_0 converges to $p - E_F(h)$, where

the spectral projection is computed in $B(pH)$. Let $\overline{B} = \{a \in A : ap = pa\}$, and let $\overline{B}_0$ be the inverse image of B_0 in $\overline{B}$. If q is the limit in $B(H)$ of an approximate identity of $\overline{B}_0$, then q is an open projection in A^{**}, $qp = pq$, and $qp = p - E_F(h)$. Thus $E_F(h)$ is $p \wedge (1-q)$, a closed projection in A^{**}, and h is in $QC(p)$.

Conversely, if $x \in QC(p)$ and $b \in B$, then $x = p\overline{x}$ and $b = p\overline{b}$ where $\overline{x} \in M(A)$, $\overline{x}p = p\overline{x}$, and $\overline{b} \in \overline{B}$. Then $xb = p\overline{x}\overline{b} \in B$ and $bx = p\overline{b}\overline{x} \in B$. Thus $x \in M(B)$.

Remark. The σ-unitality was used only in the second part of the proof.

Theorem 3.3. *If A in 3.1 is σ-unital, then the following conditions are equivalent to 1-5 of 3.1:*

6. *$pAp \subset QC(p)$.*
7. *$pM(A)p = QC(p)$.*

Proof. That $2 \Rightarrow 6$ is obvious.

$6 \Rightarrow 3$: Let x be in pAp and let a be in A. Write $x = p\overline{x}$ where $\overline{x} \in M(A)$ and $\overline{x}p = p\overline{x}$. Then $xpap = p\overline{x}pap = p^2\overline{x}ap \in pAp$.

That 7 implies 6 is obvious.

$2 \Rightarrow 7$: Clearly we have the non-degeneracy required for 3.2. Let x be in pAp and let y be in $M(A)$. Write $x = p\overline{x}$ where $\overline{x}$ is in A and $p\overline{x} = \overline{x}p$. Then $xpyp = p(\overline{x}y)p \in pAp$, and $pypx = p(y\overline{x})p \in pAp$. Thus, in the notation of 3.2, $pyp \in M(B)$, and hence $pyp \in QC(p)$.

Example 3.4. In this example p is closed, infinite rank, abelian, and atomic, and pA^*p is norm separable. Also $SQC(p) = \{0\}$ but p satisfies (MQC). In particular, p is a counterexample for the question raised in Section 0 about isolated points. In fact, if p_0 is a minimal projection, $p_0 \le p$, and $p - p_0$ is closed, then obviously $p_0 \in SQC(p)$.

Let $A = C([0,1]) \otimes \mathcal{K}$. Here $\mathcal{K}$ is the algebra of compact operators on a separable infinite dimensional Hilbert space H, $\{e_1, e_2, \dots\}$ is an orthonormal basis of H, and P_n is the projection on span$\{e_1, \dots, e_n\}$. A criterion for weak semicontinuity from [11, §5.G] will be used to describe closed projections in A^{**}. A closed projection is given by a projection-valued function $P : [0,1] \to B(H)$ such that if h is any weak cluster point of $P(y)$ as $y \to x$, then $h \le P(x)$. More precisely, P describes the atomic part of a closed projection p, and P determines p since a closed projection is determined by its atomic part. (In our case p will equal its atomic part.) We will construct a countable subset S of $[0,1]$ and unit vectors $v(x)$ for each x in S. For x in S, $P(x)$ is the rank one projection on $\mathbb{C}v(x)$, and for x not in S, $P(x) = 0$.

The following trivial lemma is stated for purposes of reference:

3.4.1. *Let $\{x_i\}$ be a sequence of distinct points in $[0,1]$ and let D be a countable subset of $[0,1]$. Then there are distinct points y_{ij} in $[0,1] \setminus (\{x_i\} \cup D)$ such that $|y_{ij} - x_i| \le 2^{-(i+j)}$.*

We will take $S = \bigcup_0^\infty S_n$, a disjoint union, where S_n and $v|_{S_n}$ will be constructed recursively so that $\|P_n v(x)\| \le n^{-\frac{1}{2}}$ for x in S_n.

Step 0: Take $S_0 = \{\frac{1}{2}\}$, $v(\frac{1}{2}) = e_1$.

Step 1: Take $S_1 = \{x_i\}$ where the x_i's are distinct, $x_i \neq \dfrac{1}{2}$, and $x_i \to \dfrac{1}{2}$ as $i \to \infty$. Let $v(x_i) = 2^{-\frac{1}{2}}e_1 + 2^{-\frac{1}{2}}e_{i+1}$ for $i = 1, 2, \dots$.

$\vdots$

Step n ($n > 1$, step $n-1$ already completed): Write $S_{n-1} = \{x_1, x_2, \dots\}$. Choose y_{ij}'s as in 3.4.1 with $D = \cup_0^{n-2} S_k$. Let $S_n = \{y_{ij} : i, j = 1, 2, \dots\}$ and $v(y_{ij}) = n^{-\frac{1}{2}}v(x_i) + (1-n^{-1})^{\frac{1}{2}}w_{ij}$, where w_{ij} is a unit vector such that $(w_{ij}, v(x_i)) = 0$ and $P_{i+j+n}w_{ij} = 0$.

The first step in the proof is to show that we get a closed projection. Thus we may assume given a sequence (t_r) in $[0,1]$ such that $t_r \to t$ and $P(t_r) \xrightarrow{w} h$. We must show $h \leq P(t)$. We have no difficulty if $P(t_r) = 0$. Thus we may assume, after passing to a subsequence, that $t_r \in S_{n(r)}$. If $n(r) \to \infty$, then since $\|P_{n(r)}P(t_r)P_{n(r)}\| \leq n(r)^{-1}$, we must have $h = 0$. Thus, after again passing to a subsequence, we may assume $n(r) = n$, $\forall r$. Now it is easy to see by induction that $\bigcup_0^n S_k$ is closed. In fact, every cluster point of S_n is in $\overline{S_{n-1}} = \bigcup_0^{n-1} S_k$. The proof that $h \leq P(t)$ will be left to the reader in the cases $n = 0$, $n = 1$. If $n > 1$, write $t_r = y_{i(r)j(r)}$ in the notation of step n. If $i(r) + j(r) \not\to \infty$, we may assume, after passing to a subsequence, that $t_r = t$, $\forall r$, a trivial case. If $i(r) + j(r) \to \infty$, then $t \in S_m$ for some $m < n$. We use induction on $n - m$. First suppose $i(r) \not\to \infty$. Then, passing to a subsequence, we assume $i(r) = i$, $\forall r$. Then $t = x_i$, and the construction shows that $h = n^{-1}P(x_i)$. If $i(r) \to \infty$, let $t'_r = x_{i(r)}$. Then $t'_r \to t$, and $[v(t_r) - n^{-\frac{1}{2}}v(t'_r)] \xrightarrow{w} 0$. This shows that $h = n^{-1}h'$, where $P(t'_r) \xrightarrow{w} h'$. Since $h' \leq P(t)$ by induction, we conclude that $h \leq P(t)$, as desired.

Now since A is separable, every state in $F(p)$ is the resultant of a probability measure on $F(p) \cap P(A)$. Since $F(p) \cap P(A)$ is countable, the integral is a Bochner integral and thus the resultant is an atomic state. This shows that p is atomic, as claimed. Also, pA^*p is norm separable, being isometrically isomorphic to $\ell^1(S)$. That p is abelian, in other words $pA^{**}p$ is abelian, is now obvious (cf [10]).

Now if h is in $pA^{**}p$, h is determined by a function λ in $\ell^\infty(S)$, where $h(x) = \lambda(x)P(x)$, $x \in S$. If h is in $SQC(p)$, then $h = p\overline{h}$, where $\overline{h} \in A$ and $p\overline{h} = \overline{h}p$. In particular, $\overline{h}(\cdot)$ is a norm continuous function from $[0,1]$ to $\mathcal{K}$. If $x \in S$, there is a sequence (x_n) in S such that $x_n \to x$ and $P(x_n) \xrightarrow{w} tP(x)$, where $0 < t < 1$. Since $P(\cdot)\overline{h}(\cdot)P(\cdot) = \lambda(\cdot)P(\cdot)$, we conclude that $\lambda(x_n) \to t\lambda(x)$. Since also $h^2 \in SQC(p)$, we also have $\lambda(x_n)^2 \to t\lambda(x)^2$. This implies $\lambda(x) = 0$. Since x is arbitrary, $h = 0$. (The only property of h actually used is that h, $h^2 \in pAp$.)

Finally we note that any continuous function on $[0,1]$ gives rise to an element $\overline{h}$ of the center of $M(A)$. Thus $p\overline{h} \in QC(p)$. It is easy to see that such elements of $QC(p)$ generate $pA^{**}p$ as a W^*-algebra, and hence p satisfies (MQC).

Example 3.5. By modifying the previous example, we can obtain either of the following:

(a) a compact projection $\tilde{p}$ such that $QC(\tilde{p}) = \mathbb{C}\tilde{p}$
(b) a closed projection p_1 such that $SQC(p_1) = \{0\}$ and $QC(p_1) = \mathbb{C}p_1$.

In both cases we will still have p infinite rank, abelian, and atomic and pA^*p

norm separable, and of course A will still be separable.

(a) Let A and p be as in 3.4, and consider $\widetilde{A}$ and $\tilde{p} = p + p_\infty$. $\widetilde{A}^{**}$ is identified with $A^{**} \oplus \mathbb{C}$ and p_∞ has its usual meaning, so that $p_\infty = 0 \oplus 1$ in $A^{**} \oplus \mathbb{C}$ and $\tilde{p} = p \oplus 1$. Then $\tilde{p}$ is closed, and hence compact, in $\widetilde{A}^{**}$. Suppose $x = \lambda 1_{\widetilde{A}} + a$, $\lambda \in \mathbb{C}$, $a \in A$, and $x\tilde{p} = \tilde{p}x$. Then $ap = pa$, and hence by 3.4, $ap = 0$. It follows easily that $\tilde{p}x = \lambda\tilde{p}$. Therefore $QC(\tilde{p}) = \mathbb{C}\tilde{p}$.

(b) We will use a C^*-algebra A_1 which is a maximal hereditary C^*-subalgebra of $\widetilde{A}$. Let p_0 be the minimal projection in $\widetilde{A}^{**}$ (actually in A^{**}) corresponding to the projection $P(\frac{1}{2})$ in the notation of 3.4 (p_0 corresponds to the pure state φ_0 where $\varphi_0(a) = (a(\frac{1}{2})e_1, e_1)$). Then $p_0 \le p \le \tilde{p}$. Let $A_1 = \mathrm{her}(1 - p_0)$ and $p_1 = \tilde{p} - p_0$ in A_1^{**}. Then A_1^{**} is identified with $(1 - p_0)\widetilde{A}^{**}(1 - p_0)$. It is easy to see that p_1 is closed in A_1^{**}: The complementary projection to p_1 in A_1^{**} is $1 - p_0 - p_1 = 1 - \tilde{p}$, and this supports a hereditary C^*-subalgebra of $\widetilde{A}$ which happens to be contained in A_1 also. If x is in A_1 and $xp_1 = p_1x$, then x is also in $\widetilde{A}$ and $x\tilde{p} = \tilde{p}x$. Thus by (a), $\tilde{p}x = \lambda\tilde{p}$ and hence $p_1x = \lambda p_1$. But $x \in A_1$ implies $xp_0 = p_0x = 0$. Since $\tilde{p}x = \lambda\tilde{p}$ implies $p_0x = \lambda p_0$, $\lambda = 0$. Therefore $SQC(p_1) = \{0\}$.

Now A_1 can be regarded as the set of all norm continuous functions $f : [0, 1] \to \widetilde{\mathcal{K}}$ such that $f(\frac{1}{2})P(\frac{1}{2}) = P(\frac{1}{2})f(\frac{1}{2}) = 0$ and the image of f in $\widetilde{\mathcal{K}}/\mathcal{K}$ is constant. Since $\frac{1}{2}$ is not an isolated point of $[0, 1]$, $M(A_1)$ can be regarded as a set of functions $g : [0, 1]\backslash\{\frac{1}{2}\} \to \widetilde{\mathcal{K}}$ (cf. [7, Theorem 3.3] and note that $\widetilde{\mathcal{K}}$ is unital). The requirements on g are:

(i) g is norm continuous and bounded.

(ii) $\lim_{t\to\frac{1}{2}} (1_{\widetilde{\mathcal{K}}} - P(\frac{1}{2}))g(t)(1_{\widetilde{\mathcal{K}}} - P(\frac{1}{2}))$ exists in norm.

(iii) $\lim_{t\to\frac{1}{2}} \|P(\frac{1}{2})g(t)(1_{\widetilde{\mathcal{K}}} - P(\frac{1}{2}))\| = \lim_{t\to\frac{1}{2}} \|(1_{\widetilde{\mathcal{K}}} - P(\frac{1}{2}))g(t)P(\frac{1}{2})\| = 0.$

(iv) If we write $g(t) = \lambda(t)1_{\widetilde{\mathcal{K}}} + x(t)$, $\lambda(t) \in \mathbb{C}$, $x(t) \in \mathcal{K}$, then $\lambda(\cdot)$ is a constant.

To see these, the main thing to note is that the constant function $1_{\widetilde{\mathcal{K}}} - P(\frac{1}{2})$ is in A_1.

Now assume g, as above, commutes with p_1. Then $x(t)$ commutes with $P(t)$ for all t in $S \setminus \{\frac{1}{2}\}$, in the notation of 3.4. Just as in 3.4, this implies $P(t)x(t) = 0$ for t in $S \setminus \{\frac{1}{2}\}$; i.e., $p_1x = 0$ and $p_1g = \lambda p_1$. Thus $QC(p_1) = \mathbb{C}p_1$.

Example 3.6. Here we show, by a simpler example, how badly Theorem 3.2 can fail when the non-degeneracy hypothesis is eliminated. By [7, Theorem 2.7], if B is a non-unital separable C^*-algebra, then $M(B)$ is non-separable. Thus if A is separable and $SQC(p)$ is non-unital (in particular non-trivial), and if the conclusion of 3.2 is true, then $QC(p)$ is much larger than $SQC(p)$. In this example, $SQC(p)$ is (infinite dimensional and) non-unital and $QC(p) = SQC(p) + \mathbb{C}p$.

Let $A = c \otimes \mathcal{K}$. Thus A^{**} can be identified with the set of bounded collections $\{h_n : 1 \le n \le \infty\}$, $h_n \in B(H)$. Let $v_n = 2^{-\frac{1}{2}}e_1 + 2^{-\frac{1}{2}}e_{n+1}$, $n < \infty$, $v_\infty = e_1$, let p_n be the projection with range $\mathbb{C}v_n$, and let $p = \{p_n\}$ in A^{**}. Then p is closed since $p_n \xrightarrow{w} \frac{1}{2}p_\infty$, and clearly p is abelian. Any element of $pA^{**}p$ is given by $h_n = \lambda_n p_n$, $1 \le n \le \infty$, $\{\lambda_n\}$ bounded. An easy argument, which is part of 3.4, shows that if

$h \in SQC(p)$ then $\lambda_\infty = 0$ and $\lambda_n \to 0$ as $n \to \infty$. Conversely, any such h is in $SQC(p)$; in fact $h \in A \cap pA^{**}p$. Thus $SQC(p) \cong c_0$. Next we show that $h \in QC(p)$ implies $\lambda_n \to \lambda_\infty$. If this is false for $h = h^*$, then there is a closed subset F of $\mathbb{R}$ such that $\lambda_\infty \notin F$ and $\lambda_n \in F$ for infinitely many n. If $q = E_F(h)$, then $q_\infty = 0$ and $q_n = p_n$ for infinitely many n. Since $p_n \to \frac{1}{2}p_\infty \neq 0$, q is not closed and h is not q-continuous on p. Thus $QC(p) \cong c$ and $QC(p)/SQC(p)$ is one dimensional.

4. Closed faces with (NCEB).

If $\widehat{A}$ is the spectrum of A and p is a projection in A^{**}, we will denote by X the set of all $[\pi]$ in $\widehat{A}$ such that $\pi^{**}(p) \neq 0$. For $[\pi]$ in X let $p_{[\pi]}$ be the atomic projection in A^{**} corresponding to $\pi^{**}(p)$. Thus $z_{\mathrm{at}}p = \sum_{x \in X} p_x$. If p is closed, or even universally measurable, then p is determined by the p_x's. If φ and ψ are in $(0,\infty)P(A)$, we will say that φ and ψ are *equivalent*, and write $\varphi \sim \psi$, if the pure states $\dfrac{\varphi}{\|\varphi\|}$ and $\dfrac{\psi}{\|\psi\|}$ are equivalent.

The proof of the next theorem and some of the other geometric arguments in this paper were inspired by Glimm [16].

Theorem 4.1. *If p is a projection in A^{**} and if p satisfies (NCEB), then p_x is finite rank, $\forall x \in X$.*

Proof. Let π be an irreducible representation belonging to x, and let H_x be the range of $\pi^{**}(p)$. If the conclusion is false, there is an infinite orthonormal sequence, $\{e_1, e_2, \dots\}$, in H_x. Choose $t > 0$ such that $[P(A) \cap F(p)]^- \subset [t,1]P(A) \cup \{0\}$ and choose s such that $0 < s < t$. Let $v_n = s^{1/2}e_i + (1-s)^{1/2}e_n$, $n > 2$, where i is 1 or 2. Define φ_n, ψ_n in $P(A) \cap F(p)$ by $\varphi_n(a) = (\pi(a)v_n, v_n)$, $\psi_n(a) = (\pi(a)e_n, e_n)$. Let θ be any cluster point of (ψ_n) in $Q(A)$. Since $e_n \xrightarrow{w} 0$, $(\pi(a)e_i, e_n) \to 0$, $\forall a \in A$. Therefore $s\psi_i + (1-s)\theta$ is a cluster point of (φ_n). If $\theta = 0$, we have a contradiction to (NCEB), since $0 < s < t$. Therefore $\theta \in [t,1]P(A)$. Since we must also have $s\psi_i + (1-s)\theta \in [t,1]P(A)$, it follows that $\theta = r_i\psi_i$ for some $r_i \geq t > 0$. We have shown that $\theta = r_1\psi_1$ and $\theta = r_2\psi_2$, a contradiction.

For the rest of this section we assume that p is closed and satisfies (NCEB). Let $\widetilde{X} = [P(A) \cap F(p)]^- \setminus \{0\}$. Then $\widetilde{X} \subset F(p) \cap [t,1]P(A)$ and $\widetilde{X}$ is locally compact, since $\widetilde{X} \cup \{0\}$ is closed. We identify X with the set of equivalence classes in $\widetilde{X}$ via $f : \widetilde{X} \to X$, where $f(\varphi) = [\pi_\varphi]$. Give X the quotient topology arising from f.

Lemma 4.2. *f is a closed map.*

Proof. The main point is to show the following: If $(\varphi_i)_{i \in D}$ and $(\psi_i)_{i \in D}$ are nets in $\widetilde{X}$ such that $\varphi_i \sim \psi_i$, $\varphi_i \to \varphi$, and $\psi_i \to \psi$, then either $\varphi = \psi = 0$ or $\varphi, \psi \in \widetilde{X}$ and $\varphi \sim \psi$. Assume this is false and consider first the case $\varphi = 0$, $\psi \in \widetilde{X}$. Let π be the reduced atomic representation of A, $H = H_\pi$, and choose vectors u_i, v_i in $\pi^{**}(p)H$ of norm at most 1 such that $\varphi_i = (\pi(\cdot)u_i, u_i)$, $\psi_i = (\pi(\cdot)v_i, v_i)$. If $g_i(a) = (\pi(a)u_i, v_i)$, then $|g_i(a)| \leq \|\pi(a)u_i\| = \varphi_i(a^*a)^{1/2} \to 0$. Therefore $g_i \to 0$. Now choose r_i in $\mathbb{R}$ such that $\|w_i\| = 1$, where $w_i = r_iu_i + (\frac{t}{2})^{1/2}v_i$. Since

$\|u_i\|^2 \geq t$, $\{r_i\}$ is bounded. Let $\theta_i = (\pi(\cdot)w_i, w_i)$. It follows from the above that $\theta_i \in F(p) \cap P(A)$ and $\theta_i \to \frac{t}{2}\psi$. Since $0 < \|\frac{t}{2}\psi\| < t$, this contradicts (NCEB).

Next assume $\varphi, \psi \in \widetilde{X}$ and $\varphi \not\sim \psi$. Then there are invariant subspaces H_1 and H_2 of H, corresponding to inequivalent irreducible representations, and non-zero vectors u in H_1, v in H_2 such that $\varphi = (\pi(\cdot)u, u)$ and $\psi = (\pi(\cdot)v, v)$. Let u_i, v_i, and g_i be as above with the extra condition that $Re(u_i, v_i) \geq 0$. Passing to a subnet, we may assume $g_i \to g$, $g \in A^*$. Since $|g_i(a)| \leq \varphi_i(a^*a)^{1/2}$, $\forall a \in A$, then $|g(a)| \leq \varphi(a^*a)^{1/2}$. From the Hahn-Banach and Riesz-Fisher theorems we see that $g = (\pi(\cdot)u, u')$ for some u' in H. Clearly, we may assume $u' \in H_1$. Similarly, $|g_i(a)| \leq \psi_i(aa^*)^{1/2}$, and hence $|g(a)| \leq \psi(aa^*)^{1/2}$. Therefore $g = (\pi(\cdot)v', v)$ for some v' in H_2. It follows that $g = 0$ ([20]). Now choose r_i in $\mathbb{R}_+$ such that $\|w_i\| = 1$, where $w_i = r_i(u_i + v_i)$. Since $2t \leq \|u_i + v_i\|^2 \leq 4$, $\{r_i\}$ is bounded and bounded away from 0. If $\theta_i = (\pi(\cdot)w_i, w_i)$, then $\theta_i \in F(p) \cap P(A)$ and every cluster point of (θ_i) is of the form $r^2(\varphi + \psi)$ for some cluster point r of (r_i). Since this last functional is not a multiple of a pure state, this contradicts (NCEB).

To complete the proof of the lemma, we have to show that the saturation of a closed set is closed. Suppose Y is a closed subset of $\widetilde{X}$, $\varphi_i \in f^{-1}(f(Y))$, and $\varphi_i \to \varphi$ in $\widetilde{X}$. Choose ψ_i in Y such that $\varphi_i \sim \psi_i$. Passing to a subnet if necessary, we may assume $\psi_i \to \psi$. By what has already been proved $\psi \in \widetilde{X}$ and $\psi \sim \varphi$. Since Y is closed, $\psi \in Y$ and hence $\varphi \in f^{-1}(f(Y))$. Thus $f^{-1}(f(Y))$ is closed (relative to $\widetilde{X}$).

Theorem 4.3. *X is a locally compact Hausdorff space and f is a proper map from $\widetilde{X}$ to X.*

Proof. The fibers of f, i.e., the sets $f^{-1}(\{x\})$, $x \in X$, are compact (even norm compact) by 4.1. This, 4.2, and the fact that $\widetilde{X}$ is locally compact Hausdorff imply that X is locally compact Hausdorff, by standard point set topology. Any closed map with compact fibers is proper; i.e., the inverse image of a compact set is compact.

Remarks. 1. It follows from 4.3, or it could be deduced directly from the proof of 4.2, that the saturation of a compact subset of $\widetilde{X}$ is compact.

2. The topology of X is stronger than, and in general unequal to, the relative topology that X inherits from the usual hull-kernel topology of $\widehat{A}$. In fact, using [5] and 1.4, we can easily construct a closed projection satisfying (NCEB) and even (CEB) such that X is a countably infinite discrete space and the image of X in prim A consists of one point. Thus the relative topology is trivial on X.

Lemma 4.4. *Assume p is an atomic closed projection satisfying (NCEB) and that pA^*p is norm separable. Then for every closed subset S of X, $\sum_{x\in S} p_x$ is a closed projection.*

Proof. Since pA^*p has a linear subspace isometric to $\ell^1(X)$, X must be countable. Let $p_S = \sum_{x\in S} p_x$. Then every element of $F(p_S)$ is the resultant of a probability measure supported by $[F(p_S) \cap P(A)] \cup \{0\}$, and *a fortiori* supported by $f^{-1}(S) \cup \{0\}$. Since $f^{-1}(S) \cup \{0\}$ is compact, every element of $F(p_S)^-$ is the resultant of a probability measure on $f^{-1}(S) \cup \{0\}$. Since $f^{-1}(S)$ is the disjoint union of

countably many fibers of f, since each of these fibers is contained in $F(p_x)$ for some x in S, and since each p_x is finite rank and hence closed, it is easy to see that any such resultant is in $F(p_S)$. Thus $F(p_S)$ is closed and hence p_S is closed.

Corollary 4.5. *Under the same assumptions, if $p \neq 0$, there is a minimal projection p_0 such that $p_0 \leq p$ and $p - p_0$ is closed. Also for every non-zero closed subprojection p' of p, there is a minimal projection p_0 such that $p_0 \leq p'$ and $p' - p_0$ is closed.*

Proof. Since X is countable and locally compact Hausdorff, the Baire category theorem implies that X has an isolated point x_0. Let p_0 be any minimal subprojection of p_{x_0}. Then $p - p_0 = (p_{x_0} - p_0) + p_{X\setminus\{x_0\}}$, the sum of two orthogonal closed projections. Therefore $p - p_0$ is closed ([1, Theorem II.7]).

Remarks. 1. In Section 6 we will generalize 4.4 and 4.5 by dropping the requirement that pA^*p be norm separable, but we will add the assumption that A is separable. We are not sure what technical assumptions are really needed.

2. Corollary 4.5 and Examples 3.4 and 3.5(a) constitute our results on the "isolated point" question raised in Section 0. The second sentence of 4.5 is closely analogous to the definition of a scattered topological space and less closely analogous to the definition of scattered C^*-algebras. Obviously we have not found a necessary and sufficient condition for this to hold. Example 3.4 shows that we cannot replace (NCEB) by the weaker condition $[F(p) \cap P(A)]^- \subset [0, 1]P(A)$, and 3.5(a) shows we cannot weaken (NCEB) to $[F(p) \cap P(A)]^- \subset \{0\} \cup [t, 1]S(A)$. Any closed face of a a scattered C^*-algebra satisfies the conclusion of 4.5 but not necessarily the hypothesis. Example 5.12 below, whose primary purpose is something else, is a closed face satisfying the conclusion of 4.5 (the proof of this is in 7.9), but not (NCEB), and which is not isomorphic to a closed face of any scattered C^*-algebra.

We now consider the geometry of $F(p)$ in more detail.

Theorem 4.6. *If p is a closed projection satisfying (NCEB) and if $(x_i)_{i\in D}$ is a net in X converging to x, then there is a subnet $(x_j)_{j\in D}$ such that one of the following holds:*

1. We have rank $p_{x_j} = k \leq n =$ rank p_x, $\forall j$; and there are orthonormal bases $\{e^j_1, \ldots, e^j_k\}$ of range $\pi_j^{**}(p_{x_j})$ and $\{e_1, \ldots, e_n\}$ of range $\pi^{**}(p_x)$ and an $n \times k$ matrix T such that $tI_k \leq T^*T \leq I_k$ and $\forall z \in \mathbb{C}^k$, $\varphi_j(z) \to \varphi(w)$, where π_j and π are irreducible representations belonging to x_j and x, $v_j = \sum_1^k z_m e^j_m$, $v = \sum_1^n w_m e_m$, $w = Tz$, $\varphi_j(z) = (\pi_j(\cdot)v_j, v_j)$, and $\varphi(w) = (\pi(\cdot)v, v)$.

2. There is φ in $P(A) \cap F(p_x)$ such that every cluster point of (φ_j) is a multiple of φ, $\forall \varphi_j \in P(A) \cap F(p_{x_j})$.

Proof. If rank $p_{x_i} \not\to \infty$, we first pick a subnet such that rank $p_{x_j} = k$, $\forall j$. If rank $p_{x_i} \to \infty$, we must show there is a subnet satisfying 2; and we do this by contradiction. Thus assume there are a subnet (x_j) and pure states θ_j, ψ_j in $F(p_{x_j})$ such that (θ_j) and (ψ_j) converge to non-proportional elements of $F(p_x)$. In the first case choose an arbitrary orthonormal basis $\{e^j_1, \ldots, e^j_k\}$ of range $\pi^{**}(p_{x_j})$. In the second case let $k = n+1$ and choose an orthonormal set $\{e^j_1, \ldots, e^j_k\}$ in range

$\pi^*(p_{x_j})$ such that $\theta_j = (\pi_j(\cdot)v_j, v_j)$ and $\psi_j = (\pi_j(\cdot)v'_j, v'_j)$ with v_j, v'_j unit vectors in span $\{e^j_1, \ldots, e^j_k\}$.

In both cases define $f^j_{\ell m}$ in A^* by $f^j_{\ell m} = (\pi_j(\cdot)e^j_m, e^j_\ell)$, $1 \le \ell, m \le k$. Passing to a subnet, we may assume $f^j_{\ell m} \to f_{\ell m}$, $\forall \ell, m$. Since the matrix $[f^j_{\ell m}]$ represents a positive linear functional on $A \otimes M_k$, the same must be true of the matrix $[f_{\ell m}]$. The GNS representation of $A \otimes M_k$ induced by $[f_{\ell m}]$ must be of the form $\tilde{\pi} \otimes id$ for some respresentation $\tilde{\pi}$ of A, and $[f_{\ell m}]$ must be the vector state induced by a vector $(u_1, \ldots, u_k)$ in $H_{\tilde{\pi}} \oplus \cdots \oplus H_{\tilde{\pi}}$. In other words, $f_{\ell m} = (\tilde{\pi}(\cdot)u_m, u_\ell)$. Since $f_{\ell\ell} \in [t, 1][P(A) \cap F(p_x)]$, $\tilde{\pi} \cong \pi \oplus \cdots \oplus \pi$. Thus we may write $u_\ell = (u_{\ell 1}, \ldots, u_{\ell r})$, $r \le k$, where $u_{\ell p} \in$ range $\pi^{**}(p_x)$.

Now $f_{\ell\ell} = \sum_1^r (\pi(\cdot)u_{\ell p}, u_{\ell p})$. Since $f_{\ell\ell} \in [t, 1]P(A)$, there must be a non-zero vector y_ℓ in range $\pi^{**}(p_x)$ such that $u_{\ell p} = \lambda_{\ell p} y_\ell$ with $(\lambda_{\ell \cdot})$ a non-zero element of $\mathbb{C}^r$. If $z \in \mathbb{C}^k$ and $\varphi_j(z)$ is as above, then $\varphi_j(z) = \sum \bar{z}_\ell f^j_{\ell m} z_m$ and hence $\varphi_j(z) \to \sum \bar{z}_\ell f_{\ell m} z_m = (\tilde{\pi}(\cdot) \sum z_\ell u_\ell, \sum z_\ell u_\ell)$. Since this functional is a multiple of a pure state, the vectors $\sum z_\ell u_{\ell p}$, $1 \le p \le r$, must be proportional. Suppose, for example, that y_1 and y_2 are linearly independent. Then the choice $z = (1, 1, 0, \ldots, 0)$ shows that $(\lambda_{1\cdot})$ and $(\lambda_{2\cdot})$ are proportional. For $\ell > 2$, y_ℓ cannot be a multiple of both y_1 and y_2. Therefore all $(\lambda_{\ell\cdot})$ are proportional. Changing notation, we may write $u_{\ell p} = \lambda_p y_\ell$. Then $\varphi_j(z) \to (\sum_1^r |\lambda_p|^2)(\pi(\cdot) \sum z_\ell y_\ell, \sum z_\ell y_\ell)$. Now choose any orthonormal basis of range $\pi^{**}(p_x)$ and let T be the matrix of $z \to (\sum_1^r |\lambda_p|^2)^{1/2} \sum z_\ell y_\ell$. Since $t\|z\|_2^2 \le \|\lim\ \varphi_j(z)\| \le \|z\|_2^2$, we must have $tI_k \le T^*T \le I_k$. This implies $k \le n$ so that 1 is proved. The other alternative is that span$\{y_\ell\}$ is one dimensional. Then let $\varphi' = (\pi(\cdot)y_1, y_1)$ and $\varphi = \frac{\varphi'}{\|\varphi'\|}$. Since each $f_{\ell m}$ is proportional to φ, $(\varphi_j(z))$ converges to a multiple of φ, $\forall z \in \mathbb{C}^k$, and more generally every cluster point of $(\varphi_j(z_j))$ is a multiple of φ for any bounded net (z_j) in $\mathbb{C}^k$. If $k =$ rank p_{x_j}, this proves 2. In the original second case, rank $p_{x_j} \to \infty$, $k = n + 1$, this proves the contradiction that establishes 2.

We say that a C^*-algebra A satisfies (CEB) or (NCEB) if the closed projection 1 in A^{**} satifies (CEB) or (NCEB). In [17, §5] Glimm proved a necessary and sufficient condition for A to satisfy a property weaker than (NCEB), $\overline{P(A)} \subset [0, 1]P(A)$. His condition is:

(i) A is CCR,
(ii) $\widehat{A}$ is Hausdorff, and
(iii) $[\pi] \in \widehat{A}$ and dim $\pi > 1$ implies $[\pi]$ is regular.

Given (i) and (ii), (iii) can be restated as follows: If I is the ideal of A such that $\widehat{I} = \{[\pi] \in \widehat{A} : \dim \pi > 1\}$, then I is a continuous trace C^*-algebra. (See [27] for the theory of continuous trace C^*-algebras.) It is presumably an easy exercise to derive a characterization of C^*-algebras satisfying (CEB) or (NCEB) (they are equivalent for C^*-algebras) from Glimm's result. In Corollary 4.7 below we derive such a characterization instead from 4.1-4.6. The purpose is not to put this result on the record, so long after [17]. The purpose is as follows: The class of closed faces of C^*-algebras admits more varied behavior than the class of C^*-algebras. One illustration of this is the contrast between the facts on the isolated point question

for atomic closed faces of C^*−algebras and the facts on scattered C^*−algebras ([18], [19]). Another illustration is the contrast between 4.6 and 4.7. (We will show by example that all of the behavior contemplated by 4.6 really occurs.) The exercise of deriving 4.7 from 4.1 to 4.6 gives some insight into why the behavior of closed faces is more varied than that of C^*−algebras.

If A is a CCR C^*−algebra with Hausdorff spectrum, then A is isomorphic to the set of continuous sections vanishing at ∞ of a continuous field, $\mathcal{A}(x)$, $x \in \widehat{A}$, of elementary C^*−algebras. If $x_0 \in \widehat{A}$ and $\mathcal{A}(x_0)$ is one dimensional, then there is a continuous section $e(\cdot)$ such that $e(x_0) = 1_{\mathcal{A}(x_0)}$ and $e(x)$ is a projection for x in some neighborhood of x_0 ([17]). We will say A is *locally unital* at x_0 if $e(x) = 1_{\mathcal{A}(x)}$ in some neighborhood of x_0.

Corollary 4.7. *The following are equivalent for a C^*−algebra A:*

1. *A satisfies (CEB)*
2. *A satisfies (NCEB)*
3. (i) *Every irreducible representation of A is finite dimensional,*
   (ii) *$\widehat{A}$ is Hausdorff,*
   (iii) *$\forall n > 1$, $\{[\pi] : dim\ \pi = n\}$ is an open subset of $\widehat{A}$, and*
   (iv) *A is locally unital at each $[\pi]$ with $dim\ \pi = 1$.*

Remark. Condition 3(iii) says that the ideal I discussed above is the c_0 direct sum of n-homogeneous C^*−algebras for various values of n. Thus the comparison of 3 with Glimm's condition is clear.

Proof. $2 \Rightarrow 3$: (i) follows from 4.1 with $p = 1$. Since $p = 1$, $X = \widehat{A}$. Since the map from $P(A)$ to $\widehat{A}$ is open for the hull-kernel topology ([17]), the hull-kernel topology is the quotient topology; i.e., our topology on X agrees with the usual one when $p = 1$. Thus (ii) follows from 4.3. Again since the map from $P(A)$ to $\widehat{A}$ is open, if dim $\pi > 1$ and $[\pi_i] \to [\pi]$, then after passing to a subnet, we can find φ_i, ψ_i in $P(A)$ associated to π_i such that the nets (φ_i), (ψ_i) converge to distinct pure states associated to π. Thus alternative 2 of 4.6 cannot hold, and $\limsup(\dim\ \pi_i) \leq \dim\ \pi$. It is always true in a C^*−algebra that $\liminf(\dim\ \pi_i) \geq \dim\ \pi$ (but for a closed face we can have $\liminf(\text{rank}\ p_{x_i}) < \text{rank}\ p_x$). This shows (iii). If x_0, $e(\cdot)$ are as above and A is not locally unital at x_0, then we can find (x_i) such that $x_i \to x_0$ and $e(x_i) \neq 1$, $\forall i$. Then we can find φ_i in $P(A)$ associated to x_i such that $\varphi_i(e) = \frac{t}{2}$. It follows that $\|\varphi\| = \frac{t}{2}$ for any cluster point φ of (φ_i), in contradiction to (NCEB). This proves (iv).

That 1 implies 2 is obvious, and the proof that 3 implies 1 is left to the reader.

Examples 4.8. (a) We can illustrate alternative 1 of 4.6 with $A = c \otimes \mathcal{K}$. Choose k and n with $k \leq n$, $t > 0$, and an $n \times k$ matrix T such that $tI_k \leq T^*T \leq I_k$. Let $S = (1 - T^*T)^{1/2}$, a $k \times k$ matrix. Let p_∞ be the projection on span$\{e_1, \ldots, e_n\}$ and for $j < \infty$ let p_j be the range projection of $\begin{pmatrix} T \\ S \end{pmatrix}$, where the matrix is regarded as a linear isometry from $\mathbb{C}^k$ to span$\{e_1, \ldots, e_n, e_{n+j}, \ldots, e_{n+j+k-1}\}$. If $p = \{p_j : 1 \leq j \leq \infty\}$, then p is a closed projection in A^{**}, p satisfies (NCEB) ((CEB) if

$t = 1$) and 4.6.1 holds with the given matrix T. (Here we think of x_j as j and x as ∞, and $\{e_1^j, \ldots, e_k^j\}$ corresponds to the columns of $\binom{T}{S}$.)

If we want a more complicated example, say one where two different subsequences give two different matrices, we can easily modify the above. Choose $k' \le n$ and an $n \times k'$ matrix T' such that $tI_{k'} \le T'^*T' \le I_{k'}$. Let $\tilde{p}_{2j-1}$ be the above p_j, and let $\tilde{p}_{2j}$ be the above p_j constructed from T' instead of T.

(b) As a first example for alternative 2 of 4.6, consider $A_1 = \{(a_n)_1^\infty : a_n \in \widetilde{\mathcal{K}}$ and (a_n) converges to a scalar in norm$\}$. Then A_1^{**} can be identified with the set of bounded collections $\{h_n : 1 \le n \le \infty\}$ such that $h_n \in B(H) \oplus \mathbb{C}$ for $n < \infty$ and $h_\infty \in \mathbb{C}$. Choose any sequence (n_j) of positive integers and define a closed projection p in A_1^{**} by: $p = \{p_j\}$, $p_\infty = 1_{\widetilde{\mathcal{K}}}$, and p_j is a rank n_j projection in $B(H)$ for $j < \infty$. It is easy to see that p satisfies (CEB) and 4.6.2. This easy example shows that there is no restriction on rank p_{x_j} when 4.6.2 holds, but this is all that it shows.

(c) For more complicated examples, in particular examples where some subsequences satisfy 4.6.1 and others 4.6.2, we can use $A_2 = A_1 \otimes \mathcal{K}$. Then $A_2^{**} \cong A_1^{**} \overline{\otimes} B(H)$.

The construction in (a) above can also be used for A_2. Let $\tilde{p}_\infty = 1 \otimes p_\infty$ and $\tilde{p}_j = q_0 \otimes p_j$ for $j < \infty$, where the p_j's are as in (a) and q_0 is a rank 1 projection in the $B(H)$-component of $\widetilde{\mathcal{K}}^{**}$. It is easy to see that $\tilde{p}$ is closed in A_2^{**} and that $F(\tilde{p})$ is isomorphic to the closed face $F(p)$ of $(c \otimes \mathcal{K})^{**}$.

We can also construct examples of 4.6.2 using A_2. Let T be a positive $k \times k$ matrix such that $tI_k \le T^2 \le I_k$, and let u be a unit vector in $\operatorname{span}\{e_1, \ldots, e_n\}$ where k and n are arbitrary. Let $S = (1 - T^2)^{1/2}$ and define a closed projection $\tilde{p}$ in A_2^{**} by: $\tilde{p} = \{\tilde{p}_j : 1 \le j \le \infty\}$, $\tilde{p}_\infty = 1 \otimes p_\infty$ for p_∞ the projection on $\operatorname{span}\{e_1, \ldots, e_n\}$, and $\tilde{p}_j$ is the range projection of $\binom{T}{S}$ where now $\binom{T}{S}$ sends $\mathbb{C}^k$ to

$$\operatorname{span}\{e_1 \otimes u, e_2 \otimes u, \ldots, e_k \otimes u, e_1 \otimes e_{n+j}, \ldots, e_1 \otimes e_{n+j+k-1}\}.$$

Then 4.6.2 holds with φ given by $\varphi(a) = (a_\infty u, u)$. Also the columns of $\binom{T}{S}$ give an orthonormal basis $\{e_1^j, \ldots, e_k^j\}$ of range $\tilde{p}_j$, and, using the notation of 4.6.1, $\varphi_j(z) \to \|Tz\|^2\varphi$. It is easy to see that $\tilde{p}$ satisfies (NCEB).

By using the idea of the second paragraph of (a), we can construct a closed projection such that different subsequences exhibit different behavior. Some subsequences can satisfy 4.6.1, with different choices of T and k, and some can satisfy 4.6.2 with $\varphi_j(z) \to \|Tz\|^2\varphi$ for different choices of T, k, and φ.

Remark. In 4.6.2 we showed only that every cluster point of (φ_j) is a multiple of φ and did not describe which multiples arise. When rank p_{x_j} is bounded, the same methods can easily be used to construct a subnet and a positive $k \times k$ matrix T such that $tI_k \le T^2 \le I_k$ and $\varphi_j(z) \to \|Tz\|^2\varphi$.

5. Type I Closed Faces and Atomic Closed Faces

If p is a projection in A^{**}, we say that p or $F(p)$ is *type I* if $pA^{**}p$ is a type I von Neumann algebra. Clearly p is type I if and only if $c(p)$, the central cover of

p, is type I. Now $F(p)$ is the normal quasi-state space of $pA^{**}p$, and for φ in $F(p)$ the kernel of π_φ contains $(1-c(p))A^{**}$. Therefore p is type I if and only if π_φ is a type I representation for all φ in $F(p)$. (It doesn't matter whether we look at π_φ or π_φ^{**}.) Because $z_{\mathrm{at}}A^{**}$ is a type I W^*-algebra, every atomic projection is type I. However, if we also require that p be closed, or just universally measurable (say), it seems that the property of being type I may be useful.

Lemma 5.1. *Let A be a separable C^*-algebra and p a type I closed projection in A^{**}. Let μ be a probability measure on $F(p)$, and let $\pi = \int^{\oplus} \pi_\omega d\mu(\omega)$, the direct integal. Then π is a type I representation.*

Proof. Let $\varphi = \int \omega d\mu(\omega)$, the resultant of μ. Then $\varphi \in F(p)$, since $F(p)$ is closed. Therefore π_φ is type I, and π_φ is a subrepresentation of π. We claim π and π_φ have the same central support in A^{**} (i.e. π is quasi-equivalent to π_φ). Therefore π is also type I.

To see the claimed quasi-equivalence, let v_ω be the cyclic vector in H_ω produced by the GNS construction, and let $v = \int^{\oplus} v_\omega d\mu(\omega)$, a vector in H_π. Then $(\pi(a)v, v) = \varphi(a)$. For every μ-measurable subset S of $F(p)$ (μ is a Borel measure, and "μ-measurable" means measurable with respect to the completion of μ) there is a projection P_S in $\pi(A)'$ such that the corresponding subrepresentation of π is $\int_S^{\oplus} \pi_\omega d\mu(\omega)$. It is easy to see that H_π is the smallest closed invariant subspace containing $P_S v$ for all such S. Moreover the cyclic subrepresentation of π generated by $P_S v$ is equivalent to a subrepresentation of π_φ. These remarks complete the proof.

The main fact needed from direct integral theory is something that the author learned from G. W. Mackey and is expressed as a lemma. For the ideas in the proof see [23], pages 112-117, and [24], especially page 159. The basic point is that the direct integral decomposition into irreducibles of a type I representation is almost unique.

Lemma 5.2 (Mackey). *Let A be a separable C^*-algebra, let $\pi'_\cdot$ and $\pi''_\cdot$ be measurable fields of irreducible representations of A defined over standard measure spaces S' and S'', and let $\pi' = \int_{S'}^{\oplus} \pi'_s d\mu'(s)$, $\pi'' = \int_{S''}^{\oplus} \pi''_s d\mu''(s)$. Assume that $\pi'_{s'}$ is inequivalent to $\pi''_{s''}$, $\forall s' \in S'$, $\forall s'' \in S''$ and that π' and π'' are type I representations. Then π' and π'' are disjoint (i.e., their central supports in A^{**} are orthogonal).*

Lemma 5.3. *Let A be a separable C^*-algebra and p a type I closed projection in A^{**}. Assume μ and ν are positive finite measures on $F(p) \cap P(A)$ such that $\int \omega d\mu(\omega) = \int \omega d\nu(\omega)$. Let E be a saturated Borel subset (or, more generally, a saturated $(\mu+\nu)$-measurable subset) of $F(p) \cap P(A)$. Then $\int_E \omega d\mu(\omega) = \int_E \omega d\nu(\omega)$ and in particular $\mu(E) = \nu(E)$.*

Proof. Let $\varphi = \int \omega d\mu(\omega) = \int \omega d\nu(\omega)$, $\pi' = \int^{\oplus} \pi_\omega d\mu(\omega)$, and $\pi'' = \int^{\oplus} \pi_\omega d\nu(\omega)$. As in the proof of 5.1, there are vectors v' in $H_{\pi'}$ and v'' in $H_{\pi''}$ which induce the functional φ. Thus there is a partial isometry U which intertwines π' and π'' such that v' is in the initial space of U and $Uv' = v''$.

Let P'_E and P''_E be the projections in $\pi'(A)'$ and $\pi''(A)'$ defined from E. Thus $\mu(E) = (P'_E v', v')$ and $\nu(E) = (P''_E v'', v'')$. By 5.2 and 5.1, $(1-P''_E)UP'_E = P''_E U(1-$

$P'_E) = 0$. Therefore $P'_E v'$ is in the initial space of U and $UP'_E v' = P''_E v''$. The conclusion follows.

Lemma 5.4. *Let A be a separable C^*-algebra, p a type I closed projection in A^{**}, and E a saturated Borel subset of $F(p) \cap P(A)$. Then there is a projection p_E in A^{**} such that $p_E \leq p$ and $F(p_E)$ is the set of resultants of probability measures on $E \cup \{0\}$.*

Proof. Let F_1 be the set of resultants of probability measures on $E \cup \{0\}$. We claim that F_1 is a norm closed sub-face of $F(p)$. The result then follows from [15, Theorem 4.4 and p. 396] (cf. [25, 3.6.11]).

To see the claim, note that by Choquet theory every element of $F(p)$ is the resultant of a probability measure on $[F(p) \cap P(A)] \cup \{0\}$. Let $E' = [F(p) \cap P(a)] \setminus E$. Then $F_1 = \{\int \omega d\mu(\omega) : \mu(E') = 0\}$. Suppose $\varphi_i = \int \omega d\mu_i(\omega)$, $i = 1, 2$, and $t\varphi_1 + (1-t)\varphi_2 \in F_1$, $0 < t < 1$. By 5.3, $t\mu_1(E') + (1-t)\mu_2(E') = 0$. Therefore $\mu_1(E') = \mu_2(E') = 0$, and $\varphi_1, \varphi_2 \in F_1$. Thus F_1 is a face.

To see that F_1 is norm closed, assume $\varphi = \int \omega d\mu(\omega)$ where $\mu(E') = \delta > 0$. We claim that $\text{dist}(\varphi, F_1) \geq \delta$. Suppose $\psi = \int \omega d\nu(\omega)$ where $\nu(E') = 0$ and $\|\varphi - \psi\| = r$. Then $\varphi - \psi = \lambda_1 - \lambda_2$ where $\lambda_1, \lambda_2 \geq 0$ and $\|\lambda_1\| + \|\lambda_2\| = r$. If $\lambda_i = \int \omega d\mu_i(\omega)$, for positive measures μ_1, μ_2 on $F(p) \cap P(A)$, then $\mu + \mu_2$ and $\nu + \mu_1$ have the same resultant. Therefore by 5.3, $\mu(E') + \mu_2(E') = \nu(E') + \mu_1(E')$. Therefore $\mu(E') \leq \mu_1(E') \leq r$. This proves the claim and completes the proof of the lemma.

Remarks. Although the conclusion of 5.4 has what we need, more is true. Also $F(p_E) \cap S(A)$ is a split face of $F(p) \cap S(A)$, the complement being $F(p_{E'}) \cap S(A)$. This means that p_E and $p_{E'}$ are centrally disjoint projections and $p_E + p_{E'} = p$. Also p_E satisfies the barycenter formula. (The barycenter formula is discussed below before 5.13). A related statement is that $F(p_E)$ is closed under resultants. The hypotheses of 5.4 could be weakened. We could assume that p satisfies the barycenter formula instead of that p is closed, and we could assume E universally measurable instead of Borel.

Lemma 5.5. *Let A be a separable C^*-algebra and p a closed projection in A^{**}. If $\pi^{**}(p)$ has finite rank for every irreducible representation π of A, then p is type I.*

Proof. Let $\pi = \int^{\oplus} \pi_s d\mu(s)$ be a standard direct integral, where each π_s is irreducible. Since p is closed, $\pi^{**}(p) = \int^{\oplus} \pi_s^{**}(p) d\mu(s)$, where $\pi_\cdot^{**}(p)$ is a Borel operator field. Therefore rank $(\pi_\cdot^{**}(p))$ is a Borel function, and by hypothesis it is everywhere finite-valued.

From the above it follows that any representation of A in a separable Hilbert space can be written as a direct sum, $\pi = \bigoplus_0^\infty \pi_n$, such that $\pi_n = \int_{S_n}^{\oplus} \pi_s d\mu(s)$ and rank $(\pi_s^{**}(p)) = n$, $\forall s \in S_n$. It was shown by A. Amitsur and J. Levitzki in [9] that there is a non-commutative polynomial G_n of $2n$ variables such that G_n vanishes on M_n^{2n} but not on M_{n+1}^{2n} (cf. [21, Lemma 2], where a weaker but adequate result is proved). Also if G_n vanishes on M^{2n} for a W^*-algebra M, then M is a direct

sum of type I_k algebras for $k \le n$. Clearly G_n vanishes on $[\pi_n^{**}(p)\pi_n(A)\pi_n^{**}(p)]^{2n}$, $n > 0$, and hence by strong continuity G_n vanishes on $[\pi_n^{**}(pA^{**}p)]^{2n}$. Therefore $\pi_n^{**}(pA^{**}p)$ is type I, $\forall n$. (For $n = 0$, $\pi_0^{**}(p) = 0$). If z_n is the central support of π_n in A^{**}, and $z(\pi)$ is the central support of π, then $z(\pi) = \sup_n \; z_n$. Since we have shown that $z_n pA^{**}p$ is type I, $\forall n$, then $z(\pi)pA^{**}p$ is type I. Since $\sup\{z(\pi) : \pi \;\text{ as above}\} = 1$, $pA^{**}p$ is type I.

Corollary 5.6. *If A is a separable C^*-algebra, p is a closed projection in A^{**}, and if p satisfies (NCEB), then p is type I.*

Proof. Combine 4.1 and 5.5.

We have already defined the concept of an atomic projection in A^{**}. We say that p is *strongly atomic* if p is atomic and pA^*p is norm separable. If A is separable the separability of pA^*p can be rephrased: There are only countably many points $[\pi]$ in $\widehat{A}$ such that $\pi^{**}(p) \neq 0$.

Question 5.7. If A is separable, is every closed atomic projection in A^{**} strongly atomic?

If p is closed and atomic and if μ is a probability measure on $F(p) \cap P(A)$, then $\int \omega d\mu(\omega)$ is in $F(p)$ and hence is an atomic state. If A is separable, it follows from 5.3, for example, that μ is supported by the union of countably many equivalence classes. If p is not strongly atomic, this means that there are uncountably many equivalence classes in $F(p) \cap P(A)$ but every finite measure is concentrated on the union of countably many. It follows that the relation of equivalence of pure states is not countably separated on $F(p) \cap P(A)$. (If it were countably separated, the quotient Borel space would be an uncountable analytic Borel space ([24, Theorem 5.1]) and hence would support a continuous measure. This measure could be lifted to $F(p) \cap P(A)$ by the von Neumann selection lemma.) In particular A is not type I. Also p does not satisfy (NCEB), since the space X of Section 4 is second countable and hence countably separated when A is separable. This reasoning suggests the following:

Question 5.8. If A is a separable C^*-algebra and p is a type I closed projection in A^{**}, is equivalence of pure states countably separated on $F(p) \cap P(A)$?

Obviously 5.8 is analogous to Mackey's conjecture ([23, p. 85] or [24, p. 163]), which was proved by Glimm in [17]. Of course [17] proved much more than Mackey's conjecture. We do not know whether there is a structure theorem for type I closed faces of similar power to Glimm's theorem. Because the variety of closed faces of C^*-algebras is so great, there is not enough evidence to support a conjecture on any of these questions.

If the answer to 5.8 is yes for a particular p, then a standard form for elements of $F(p) \cap S(A)$ can be established. Let X be the set of equivalence classes of $F(p) \cap P(A)$, an analytic Borel space which is in one-one corespondence with a subset $\{[\pi_x] : x \in X\}$ of $\widehat{A}$. Then an element φ of $F(p) \cap S(A)$ is determined by a probability measure μ on X and a measurable function $f : X \to S(A)$ such that $f(x)$ is supported by $\pi_x^{**}(p)$. In fact φ is the resultant of a probability measure

$\overline{\mu}$ on $F(p) \cap P(A)$. Even though $\overline{\mu}$ is not unique, 5.3 implies its pushforward to X is unique. The function f is obtained by writing $\overline{\mu} = \int_X \nu_x d\mu(x)$, where ν_x is supported on the equivalence class x, and $f(x) = \int \omega d\nu_x(\omega)$. It can be shown that f is unique modulo null sets. Thus, under the hypotheses given, the Choquet decomposition of φ is almost unique in a sense roughly analagous to Mackey's result that the direct integral decomposition of a type I representation into irreducibles is almost unique.

There is a converse question to 5.7 which we can answer. The proof is valid even for A nonseparable.

Proposition 5.9. *If A is a C^*-algebra and p is a closed projection in A^{**} such that $z_{\mathrm{at}}pA^*p$ is norm separable, then p is atomic and hence strongly atomic.*

Proof. There is an increasing sequence (p_n) of finite rank projections such that $p_n \to z_{\mathrm{at}}p$. By 4.5.12 or 4.5.15 of [25], $z_{\mathrm{at}}p$ is universally measurable. Since $(1-z_{\mathrm{at}})p$ is a universally measurable operator whose atomic part is 0, $(1 - z_{\mathrm{at}})p = 0$ ([25, 4.3.15]).

The following lemma, or the ideas in its proof, might be useful in connection with questions 5.7, 5.8. It will also be used to prove a complement to Glimm's theorem.

Lemma 5.10. *If p is an atomic projection in A^{**} such that pA^*p is norm separable, then $F(p) \cap P(A)$ is an F_σ set relative to $P(A)$.*

Proof. The lemma can be rephrased more concretely: Let $\pi : A \to B(H)$ be an irreducible representation, let H_0 be a separable closed subspace of H, and let $P_0 = \{(\pi(\cdot)v, v) : v \text{ is a unit vector in } H_0\}$. Then P_0 is an F_σ set relative to $P(A)$.

The proof is similar to that of 4.1. Let $H_1, H_2, \ldots$ be an increasing sequence of finite dimensional subspaces such that $H_0 = (\cup_1^\infty H_n)^-$, and let p_n be the projection on H_n. Let $V_n = \{v \in H_0 : \|v\| = 1 \text{ and } \|p_n v\| \geq \frac{1}{2}\}$ and $P_n = \{(\pi(\cdot)v, v) : v \in V_n\}$. Then $P_0 = \cup_1^\infty P_n$, and we will show P_n closed relative to $P(A)$. Suppose $v_i \in V_n$, $\varphi_i = (\pi(\cdot)v_i, v_i)$, and the net (φ_i) converges to a pure state φ. Passing to a subnet if necessary, we may assume $v_i \xrightarrow{w} v$ for some v in H_0. Clearly $\|v\| \leq 1$ and $\|p_n v\| \geq \frac{1}{2}$. Then $v_i = u_i + w_i$, where $u_i \to v$ in norm, $w_i \xrightarrow{w} 0$, and $(u_i, w_i) = 0$. Therefore $(\pi(a)u_i, w_i) \to 0$, $\forall a \in A$. Passing to a further subnet, we may assume $(\pi(\cdot)w_i, w_i)$ converges to some ψ in $Q(A)$. Then $\varphi = (\pi(\cdot)v, v) + \psi$. Since φ is pure, ψ must be proportional to $(\pi(\cdot)v, v)$. Therefore $\varphi = (\pi(\cdot)v_1, v_1)$ where $v_1 = v/\|v\|$. Since $\|p_n v_1\| \geq \|p_n v\|$, $\varphi \in P_n$.

Proposition 5.11. *If A is a separable C^*-algebra and $\pi : A \to B(H)$ is an irreducible representation such that $\pi(A) \not\supset \mathcal{K}(H)$, then there are uncountably many inequivalent irreducible representations of A with the same kernel as π.*

Remark. Glimm's theorem implies that there are uncountably many irreducibles with the same kernel, but so far as we know, it was not previously known that that kernel can be taken to be the same as the kernel of the given π.

Proof. By replacing A with its quotient by the kernel of π, we may reduce to the case π faithful. Assume that A has only countably many faithful irreducible

representations. Since $\widehat{A}$ is second countable, there is a countable set $\{I_n\}$ of non-zero (closed, two-sided) ideals such that every non-faithful representation of A vanishes on some I_n. Then since $[\pi]$ is a dense point in $\widehat{A}$, the hull of I_n has empty interior in $\widehat{A}$. Let $F_n = \{\varphi \in P(A) : \varphi|I_n = 0\}$. Since the map from $P(A)$ to $\widehat{A}$ is open, we have that F_n is a closed nowhere dense set relative to $P(A)$. It now follows from the Baire category theorem, applied to $P(A)$, and 5.10 that there is a faithful irreducible representation π' whose associated pure states have non-empty interior in $P(A)$. From the openness of the map from $P(A)$ to $\widehat{A}$, we conclude that $\widehat{A}$ has an open point, whence A has an ideal K, necessarily essential, such that $\widehat{K}$ has only one point. The proof is concluded by showing $K \cong \mathcal{K}(H)$, and this can be done in at least two ways. There is a simple way to prove that every separable C^*-algebra whose spectrum is a single point must be elementary (i.e., the affirmative answer to Naimark's question in the separable case), or one can apply Glimm's theorem to K.

The following example demolishes one naive conjecture with regard to the structure of type I closed faces.

Example 5.12. If A is any non-type I separable C^*-algebra, then A has a type I closed face $F(p)$ (p is even compact) such that $F(p)$ is not isomorphic to a closed face of any type I C^*-algebra.

If A is not unital, we consider A^{**} as a subalgebra of $\widetilde{A}^{**}$ and construct p as a projection in A^{**} closed in $\widetilde{A}^{**}$, so that p will be compact. Since A is not type I, there is an irreducible representation π such that $\pi(A) \not\supset \mathcal{K}(H_\pi)$. For the natural extension of π to $\widetilde{A}$, also denoted π, we also have $\pi(\widetilde{A}) \not\supset \mathcal{K}(H_\pi)$. Let v_0 be a unit vector in H_π, $\varphi_0 = (\pi(\cdot)v_0, v_0)$ and p_0 the support projection in A^{**} of φ_0. By a result of Glimm [16, Theorem 2], there is a sequence $\{v_n\}$ of unit vectors in H_π such that $v_n \xrightarrow{w} 0$ and $(\pi(\cdot)v_n, v_n) \to \varphi_0$ in $\widetilde{A}^*$. By using the Gram-Schmidt process, we can find a subsequence $\{v_{n_i}\}$ and an orthonormal sequence $\{w_{n_i}\}$ such that $(w_{n_i}, v_0) = 0$ and $\|w_{n_i} - v_{n_i}\| \to 0$. Let $\varphi_i = (\pi(\cdot)w_{n_i}, w_{n_i})$.

Since p_0 is a minimal projection in A^{**}, it is closed in $\widetilde{A}^{**}$. Let B be the hereditary C^*-subalgebra of $\widetilde{A}$ supported by $1-p_0$, and let e be a strictly positive element of B. Since $\varphi_i \to \varphi_0$ in $\widetilde{A}^*$ and $\varphi_0|_B = 0$, $\varphi_i(e) \to 0$. Passing to a subsequence, we may assume $\sum \varphi_i(e) < \infty$. Let p_i be the support projection of φ_i. p_i is in $B^{**} \cap A^{**}$, considered as a subalgebra of $\widetilde{A}^{**}$. By 0.1(ii), $\sum_1^\infty p_i$ is closed in B^{**}. Thus if $p = \sum_0^\infty p_i$, p is closed in $\widetilde{A}^{**}$. Since $p \in A^{**}$, p is a compact projection in A^{**}. Since $pA^{**}p \cong B(H_0)$ where $H_0 = \overline{\operatorname{span}}\{v_0, w_{n_1}, w_{n_2} \dots\}$, p is a type I projection.

Suppose $F(p)$ were isomorphic to a closed face, $F(p')$, of a type I C^*-algebra A'. Since $p'(A')^{**}p'$ can be identified with the space of bounded affine functionals vanishing at 0 on $F(p')$, $p'(A')^{**}p'$ is Jordan $*-$isomorphic to $pA^{**}p$. Therefore $p'(A')^{**}p'$ is a type I factor, and p' is associated with a single irreducible representation, π', of A'. Since A' is type I, $\pi'(A') \supset \mathcal{K}(H_{\pi'})$. Let φ'_i, $i \geq 0$, be the element of $F(p')$ corresponding to φ_i. Then $\varphi'_i \to \varphi'_0$ in ${A'}^*$. This contradicts the facts that $\{\varphi'_i\}$ arises from an orthonormal sequence of vectors in $H_{\pi'}$ and $\pi'(A') \supset \mathcal{K}(H_{\pi'})$.

We think it is fairly obvious from the proof of 0.1(ii) ([12, Lemma 3]), that the faces $F(p)$ constructed above are all isomorphic. In Section 7 we will determine the structure of pAp, and this will be our formal proof of this fact.

Finally, we want to generalize 5.5 for use in connection with a remark in Section 7. If $h \in A^{**}$, we say that h satisfies the barycenter formula if, when regarded as a function on $Q(A)$, h is measurable with respect to (the completion of) any regular Borel measure and $\varphi(h) = \int h(\omega)d\mu(\omega)$ whenever μ is a regular Borel measure on $Q(A)$ and φ is the resultant of μ. If A is separable, it is sufficient to verify the formula for measures supported on $P(A)$. Also when A is separable, the barycenter formula is equivalent to: $\pi_s^{**}(h)$ is a measurable field of operators and $\pi^{**}(h) = \int^{\oplus} \pi_s^{**}(h)d\mu(s)$, whenever $\pi = \int^{\oplus} \pi_s d\mu(s)$, a standard direct integral; and again it is sufficient to verify this in the special case where each π_s is irreducible. Thus for A separable the set of elements of A^{**} satisfying the barycenter formula is a C^*-algebra closed under weak sequential convergence. This C^*-algebra is at least as large as $\{h : \text{Re } h, \text{Im } h \text{ are universally measurable}\}$ and appears to be a good thing to use, though the monotone sequential closure of A (discussed in [25, §4.5]) would do for our purposes. For A non-separable, we know of nothing more general than the space of universally measurable operators ([26]).

Theorem 5.13. *If A is a separable C^*-algebra, p is a projection in A^{**} satisfying the barycenter formula, and if $\pi^{**}(p)\pi(A)\pi^{**}(p) \subset \mathcal{K}(H_\pi)$ for every irreducible representation π of A, then p is type I.*

Proof. First note that the proof of 5.5 is equally valid if p satisfies the barycenter formula instead of being closed. Let e be a strictly positive element of A. Then for $\epsilon > 0$, $E_{[\epsilon,\infty)}(pep)$ satisfies the barycenter formula and $\pi^{**}(E_{[\epsilon,\infty)}(pep))$ has finite rank for π irreducible. Therefore each p_n is type I where $p_n = E_{[n^{-1},\infty)}(pep)$. Since $p_n \nearrow p$, p is type I.

6. More on Closed Faces with (NCEB) for A Separable.

The notations $X, p_x, \widetilde{X}$, and f have the same meanings as in Section 4.

Theorem 6.1. *If A is a separable C^*-algebra, p is a closed projection in A^{**}, and if p satisfies (NCEB), then $\sum_{x\in S} p_x$ is the atomic part of a closed projection p_S for every closed subset S of X. Also p satisfies (CEB) if and only if p_S is compact for S compact.*

Proof. By 5.6, p is type I. Let $\widetilde{S} = f^{-1}(S)$, a closed subset of $\widetilde{X}$, let $E = \widetilde{S} \cap P(A) = \widetilde{S} \cap S(A)$, a saturated subset of $F(p) \cap P(A)$, and let p_S be the projection called p_E in 5.4. By 5.4, $F(p_S) = \{\int \omega d\mu(\omega) : \mu \text{ is a probability measure on } E \cup \{0\}\} = \{\int \omega d\mu(\omega) : \mu \text{ is a probability measure on } \widetilde{S} \cup \{0\}\}$. Since $\widetilde{S} \cup \{0\}$ is a compact subset of A^*, $F(p_S)$ is closed, and hence p_S is closed by [15, Theorem 4.8]. By 5.4, $F(p_S) \cap P(A) = E$, and this implies that the atomic part of p_S is $\sum_{x\in S} p_x$.

If p satisfies (CEB) and S is compact, then $E = \widetilde{S}$, and $\widetilde{S}$ is compact by 4.3. Thus $F(p_S) \cap S(A) = \{\int \omega d\mu(\omega) : \mu \text{ is a probability measure on } \widetilde{S}\}$, a closed subset of A^*. Therefore p_S is compact. Conversely, if S compact implies p_S compact and

if $\varphi_n \to t\varphi$, for φ_n, φ in $F(p) \cap P(A)$, then there is a compact set S such that $f(\varphi_n) \in S$ for n sufficiently large. Since $F(p_S) \cap S(A)$ is closed, it follows that $t = 1$.

Remarks. 1. If p satisfies only (NCEB) and S is compact, then p_S is nearly relatively compact in the sense of [13].

2. The hypothesis of 4.4 included the assumption that p be strongly atomic, though this term was not used. Theorem 6.1 shows that this assumption can be dropped if A is separable. Also the discussion after 5.7 shows that if A is separable, p is closed and atomic, and p satisfies (NCEB), then p is strongly atomic. Thus for A separable, the hypothesis of 4.5 can be weakened by replacing strongly atomic with atomic.

3. In view of the remarks after 5.4, it is not hard to calculate the facial topology on the extreme boundary of $F(p)$, when p is closed and satisfies (CEB). Its T_0-ification is the compact Hausdorff space $X \cup \{\infty\}$. If p satisfies only (NCEB), we still see that the closed split faces of $F(p)$ containing 0 are in one-to-one correspondence with the closed subsets of X.

7. Some Relationships among Prior Sections and Concluding Remarks.

Each of the three main parts of this paper (Sections 2, 3, and 4-6) studies a different generalization of the situation considered in [5] (1.4 and 7.2 below are used to justify this statement). Sections 3-6 were motivated by our desire to investigate the circumstances in which 2.4 applies, but the detailed discussion below makes it clear that we have not solved this problem – if it can be called a "problem". There is a broader "problem" to which all three parts of this paper are relevant: Study the structure of those closed faces of C^*-algebras which are closely modeled on locally compact Hausdorff spaces. We now discuss the relationships among the prior sections.

First we consider the relationship between Sections 2 and 3. The next result and the remarks following show that if we were willing to use the theory of relative q-continuity in the construction of MASA's, it would have been sufficient to prove the special case of 2.4 in which the projection p_X is central and abelian. However, so far as we know, this special case is no easier.

Theorem 7.1. *Let A be a separable C^*-algebra and p a closed projection in A^{**}. Suppose B is a commutative C^*-subalgebra of $SQC(p)$ which is non-degenerately embedded in $pA^{**}p$. If $\widehat{B}$ is totally disconnected, then there is a commutative C^*-subalgebra C of A such that C contains an approximate identity of A, $p \in C^{**}$, and $pC = B$.*

Proof. Let $\overline{B} = \{a \in A : ap = pa \text{ and } pa \in B\}$. Then $\mathrm{her}(1-p)$ is an ideal of $\overline{B}$ and $\overline{B}/\mathrm{her}(1-p) \cong B$. We can apply 2.4 (or 2.3) with $\overline{B}$ playing the role of A and with $X = \widehat{B}$. For x in X, p_x is the support projection in $\overline{B}^{**}$ of the pure state of $\overline{B}$ given by x. Let C be the MASA of $\overline{B}$ given by 2.4. Since B is non-degenerate in $pA^{**}p$, $\overline{B}$ hereditarily generates A. Since C hereditarily generates $\overline{B}$ by 2.4, C also hereditarily generates A. One way to deduce from 2.4 that pC, which is the

image of C in $\overline{B}/\mathrm{her}(1-p)$, is all of B is to quote the classical Stone-Weierstrass theorem.

Suppose p is a closed projection in A^{**} such that $SQC(p)$ is non-degenerate in $pA^{**}p$ (cf. 3.2) and that B is a MASA in $SQC(p)$ which hereditarily generates $SQC(p)$. If A is separable and $\widehat{B}$ is totally disconnected, then 7.1 gives a commutative algebra C (which could be assumed a MASA in A). For each x in $\widehat{B}$ we have a pure state φ_x of B (or C) which is supported by a minimal projection p_x in B^{**}, and it follows from $pC = B$ and $p \in C^{**}$ that also $p_x \in C^{**}$. If p_x is minimal in A^{**}, then φ_x satisfies (UEP) relative to the inclusion of C in A. If pAp is an algebra (cf 3.1 and 7.2 below), then we need only start with a MASA B in pAp which hereditarily generates pAp and such that each pure state of B satisfies (UEP) relative to pAp. It was pointed out in Section 0 that under the hypotheses of 2.4 every element of $C_0(X)$ gives an element of $SQC(p_X)$. It can be shown that $C_0(X)$ is thus embedded as a MASA in $SQC(p_X)$ and that $C_0(X)$ is nondegenerate in $p_XA^{**}p_X$. Thus the above discussion applies.

Proposition 7.2. *Conditions (i)-(iv) of 1.2 imply that pAp is an algebra, and p satisfies:*

$$\text{(G)} \qquad [P(A) \cap F(p)]^- \subset [0,1]P(A).$$

Proof. The reduced atomic representation, π, of A is faithful on $pA^{**}p$. Moreover, $\pi^{**}(pA^{**}p) \cap \mathcal{K}(H_\pi)$ is a C^*-algebra, and by 1.2(iv), $\pi^{**}(pAp)$ is contained in this algebra. We show equality. Let h be an element of $pA^{**}_{\mathrm{sa}}p$ such that $\pi^{**}(h)$ is compact. It is sufficient to show $h \in SQC(p)$. If F is a closed subset of $\mathbb{R}$, then 1.2(i) implies that $E_F(h)$, computed in $pA^{**}p$, is closed. (In fact we don't need F closed for this.) If $0 \notin F$ (and F is closed), then $\pi^{**}(E_F(h))$ is a finite rank operator on H_π, and by [1, Corollary II.8] this implies $E_F(h)$ is compact. Thus $h \in SQC(p)$. Then (G) follows from [17, §5] for example.

The same reasoning shows $pA^{**}p = QC(p)$.

Next we consider the relationship between Sections 2 and 4. By 6.1 if A is separable and p is a closed projection in A^{**} satisfying (CEB) then we have the hypotheses of 2.3 (except for total disconnectedness of X). By 4.1 each p_x is of finite rank. For 2.4, we would want each p_x to be of rank 1. This happens for the p_x's of Section 4 if and only if p is abelian. If p is not abelian, it might be possible to write $p_x = p_{x,1} + \cdots + p_{x,n_x}$ where the $p_{x,i}$'s are minimal and $\{p_{x,i}\}$ satisfies the hypotheses of 2.4 with X replaced by some space $\overline{X}$. Then $\overline{X}$ would map onto X by a closed continuous map with finite fibers. However, Example 7.6(a) below shows that this is not always possible.

Conversely, suppose the hypotheses of 2.4 are satisfied. By 1.4, if X is countable and discrete and each equivalence class of $\{\varphi_x : x \in X\}$ is finite, then p_X satisfies (CEB). If p_X is abelian, we can deduce (7.3 below) that p_X satisfies (CEB) even for X not discrete; but it is fairly obvious (cf 7.6(b) below) that in general p_X need not satisfy (NCEB) or even (G).

In retrospect it seems that (G) is worthy of more study in the present context despite the fact, as pointed out in the remark following 4.5, that it does not imply a positive answer to the isolated point question. One reason is mentioned in 7.2 above. However, the conclusion of 1.4 is definitely false if we drop the hypothesis of finite equivalence classes. (This follows from 4.1.) It may be that (G) is part of the hypothesis of a nice result. Also, even though, by Example 3.4, (G) does not imply that $F(p)$ is associated with a locally compact Hausdorff space, we do not know whether (G) implies that $F(p)$ is associated with a Hausdorff space. We will show below that (G) does imply that p is type I.

One could also consider weaker conditions than (G):

$$[P(A)\cap F(p)]^- \subset \{\text{type I factorial quasi-states}\} \tag{1}$$

$$[P(A)\cap F(p)]^- \subset z_{\text{at}}A^*. \tag{2}$$

(2) is suggested by the theory of perfect C^*-algebras ([8]).

With regard to the relationship between Sections 3 and 4, we note that a closed projection p satisfying (CEB) need not satisfy (MSQC) (cf. 7.6(c) below). However, it follows from 6.1 that p does satisfy the hypothesis of 3.2 (A separable). Also if p is closed and abelian and satisfies (CEB), then p does satisfy (MSQC). (It follows from 5.3 that $F(p)$ is isomorphic to the set of probability measures on $\widetilde{X}\cup\{0\}$. Is there a less technical proof?) It may be that there are other useful concepts on the extensiveness of $SQC(p)$.

At the end of this section we return to Example 5.12, partly to show that it does satisfy the conclusion of 4.5. A complete theory for closed faces of C^*-algebras analogous to the theory of scattered C^*-algebras might have to be quite complicated.

Proposition 7.3. *Assume the hypotheses of 2.4 and also that p_X is abelian. Then p_X satisfies (CEB).*

Remark. The hypothesis that p_X is abelian can be rephrased more concretely: The φ_x's are mutually inequivalent. ([10]).

Proof. Since p_X is abelian, $P(A)\cap F(p_X)=\{\varphi_x : x\in X\}$. Suppose $\varphi_{x_i}\to\psi$ in $Q(A)$. Passing to a subnet, we may assume $x_i\to x$ in X or $x_i\to\infty$. If $x_i\to x$, let $\{S_j\}$ be a set of compact neighborhoods of x such that $\bigcap_j\ S_j=\{x\}$. Since p_{S_j} is compact and $\varphi_{x_i}\in F(P_{S_j})$ for i large, we conclude that $\|\psi\|=1$ and $\psi\in\bigcap_j\ F(p_{S_j})=F(\bigwedge_j\ p_{S_j})$. Since a closed projection is determined by its atomic part, $\bigwedge_j\ p_{S_j}=p_x$, and hence $\psi=\varphi_x$. If $x_i\to\infty$, let $\{U_j\}$ be a set of relatively compact open subsets of X such that $\bigcup_j\ U_j=X$, and let $S_j=X\setminus U_j$. Since x_i is eventually in S_j and since p_{S_j} is closed, we see that $\psi\in\bigcap_j\ F(P_{S_j})=F(\bigwedge_j\ p_{S_j})=\{0\}$.

It would be desirable if the hypotheses in 2.4 that certain projections are atomic parts of closed projections could be stated entirely in terms of pure states (or

equivalently, minimal projections). This can be done in a situation of intermediate generality. Consider the following conditions for a projection p in A^{**}:

$$\exists t \in (0,1] \text{ such that } [P(A) \cap F(p)]^- \subset \{0\} \cup [t,1][P(A) \cap F(p)]. \tag{3}$$

$$\{0\} \cup [P(A) \cap F(p)] \text{ is closed.} \tag{4}$$

$$[P(A) \cap F(p)] \text{ is closed.} \tag{5}$$

Conditions (3) and (4) are strengthenings of (NCEB) and (CEB) respectively, and are equivalent to (NCEB) and (CEB) if p is closed. But we are interested in the case where p is atomic. If p is the atomic part of a closed projection q, then q satisfies (NCEB) or (CEB) if and only if p satisfies (3) or (4). If p is atomic and satisfies (3) or (4), is p necessarily the atomic part of a closed projection? We can prove this if p is strongly atomic, in which case p itself is closed. (In general let C be the closed convex hull of $\{0\} \cup [P(A) \cap F(p)]$. If q exists, then $F(q) = C$. The tricky thing is to prove that C is a face of $Q(A)$.)

Lemma 7.4. *If p is a strongly atomic projection satisfying (3), then p is closed.*

Proof. Let $X = \{0\} \cup [P(A) \cap F(p)]^-$ and let C be the closed convex hull of X. Then every element of C is the resultant of a probability measure on X. Since X is norm separable, the resultant is actually a Bochner integral. Hence $C \subset F(p)$. The reverse inclusion follows easily from the structure of atomic von Neumann algebras.

Remark. The same argument works if (3) is replaced by a similar modification of (2).

Corollary 7.5. *Let X be a locally compact Hausdorff space with only countably many points, and let $\{p_x : x \in X\}$ be a family of mutually orthogonal minimal projections in A^{**}. If $\sum_{x\in S} p_x$ satisfies (3) whenever S is a closed subset of X and (5) when S is compact, and if A is separable, then we have the hypotheses and conclusions of Corollary 2.4.*

Remarks. 1. Let φ_x be the pure state supported by p_x. If $p = \sum_{x\in X} p_x$ is abelian, i.e. if the φ_x's are mutually inequivalent, then the hypotheses on $\sum_{x\in S} p_x$ can be stated more concretely:

$$\text{If } x_n \to x, \text{ then } \varphi_{x_n} \to \varphi_x, \text{ and if } x_n \to \infty, \text{ then } \varphi_{x_n} \to 0. \tag{6}$$

We can actually replace the hypotheses on $\sum_{x\in S} p_x$ by (6) if we assume only that the equivalence classes have bounded finite cardinality. (Note that they have to be finite by 4.1 if p satisfies (3).) The proof of this uses Akemann's result in [1] that the supremum of finitely many mutually commuting closed projections is closed.

2. Even when X is uncountable, the hypotheses on $\sum_{x\in S} p_x$ in 2.4 can be modified somewhat: If $\sum_{x\in X} p_x$ is the atomic part of a closed projection, and

if $\sum_{x\in S} p_x$ satisfies (3) for S closed and (5) for S compact, then we have the hypotheses of 2.4. We will not provide a complete proof of this because it would be rather technical and it is not clear that the result is a big improvement on 2.4. The main lemma is the following:
Let p be a closed projection satisfying (NCEB) and q a subprojection of $z_{\text{at}}p$. If A is separable and q satisfies (3), then q is the atomic part of a closed projection.
The proof of this uses 5.6, the other results of Section 5 (in particular the discussion following 5.8), and the von Neumann selection lemma.

Examples 7.6. (a) Let $A = c \otimes \mathcal{K}$ and define a closed projection p in A^{**} by letting p_∞ be the projection on $\operatorname{span}\{e_1, e_2\}$ and p_n the projection on

$$\begin{cases} \mathbb{C}e_1, & n = 3k+1 \\ \mathbb{C}e_2, & n = 3k+2 \\ \mathbb{C}(2^{-\frac{1}{2}}e_1 + 2^{-\frac{1}{2}}e_2), & n = 3k \end{cases}$$

It is easy to see that p satisfies (CEB). Let φ_n be the pure state of A supported by p_n, $n < \infty$, and suppose B is a MASA of A such that each $\varphi_n|_B$ satisfies (UEP). Thus each p_n is in B^{**}. If $b \in B$, then e_1 is an eigenvector of each b_{3k+1} and hence e_1 is an eigenvector of b_∞. Similarly e_2 and $2^{-\frac{1}{2}}e_1 + 2^{-\frac{1}{2}}e_2$ are eigenvectors of b_∞. Therefore all three eigenvalues are the same and $b_\infty p_\infty = \lambda p_\infty$. It follows that p_∞ is a minimal projection of B^{**}. Thus no matter how we write $p_\infty = p' + p''$, with p' and p'' rank one projections, we cannot achieve the conclusion of 2.4, let alone the hypotheses.

(b) First note that if p is the projection of 5.12, then we have the hypotheses of 2.4 with $X = \mathbb{N} \cup \{\infty\}$ and $p = p_X$. Since all of the φ_n's, $1 \le n \le \infty$, are equivalent, it is easy to see that the non-pure state $\frac{1}{2}\varphi_1 + \frac{1}{2}\varphi_\infty$ is in $[P(A) \cap F(p)]^-$ (cf [16]), so that p does not satisfy (G).

It is better to give an example where the equivalence classes of $\{\varphi_x : x \in X\}$ are finite, since by 4.1 there is no hope of (NCEB) without this finiteness. A standard example suffices for this. Let $A = \{a \in c \otimes M_2 : a_\infty \text{ is diagonal}\}$. Let $B = \{a \in A : a_n \text{ is diagonal}, \forall n\}$. Then B is a MASA in A, and we let $X = \widehat{B}$, the disjoint union of two copies of $\mathbb{N} \cup \{\infty\}$. It is clear that for x in X the pure state ψ_x of B satisfies (UEP); and if p_x is the support projection of ψ_x, we have the hypothesis of 2.4 with $p_X = 1_A$. Since $\widehat{A}$ is not Hausdorff, it follows from [17, Thm. 6] that p_X does not satisfy (G). Of course, this is also easy to see explicitly.

It is possible to give a similar example in which $\widehat{A}$ is Hausdorff, but a different condition of [17, Theorem 6] is violated. Let $A = \{a \in c \otimes M_2 \otimes M_2 : a_\infty \in M_2 \otimes I_2\}$. If $B = \{a \in A : a_n \in D_2 \otimes D_2, n < \infty; a_\infty \in D_2 \otimes I_2\}$, where $D_2 = \{d \in M_2 : d \text{ is diagonal}\}$, then B is a MASA in A and we can proceed similarly to the above. Again X is the disjoint union of two copies of $\mathbb{N} \cup \{\infty\}$ (arising more naturally as $\mathbb{N} \cup \mathbb{N} \cup \{\infty\}$).

(c) Consider one of the examples of alternative 2 of 4.6 constructed in 4.8(c). Let $k = n = 2$ and $T = 1$. We then get a projection p satisfying (CEB) where the space X of Section 4 is $\mathbb{N} \cup \{\infty\}$, rank $p_n = 2$, $1 \le n \le \infty$, and there is a single

element φ of $F(p_\infty)\cap P(A)$ such that every sequence (ψ_n) with ψ_n in $F(p_n)\cap P(A)$ converges to φ. In this case we can write $p_n = p_{n,1} + p_{n,2}$ so that the hypotheses of 2.4 are satisfied. All we have to do is take $p_{\infty,1}$ to be the support projection of φ. $\widetilde{X}$ will be homeomorphic to $\mathbb{N}\cup\{\infty\}$, but it arises as the disjoint union of $\mathbb{N}\cup\mathbb{N}\cup\{\infty\}$ with an isolated point. This example does not satisfy (MSQC). One way to see this is to note that the saturation of an open subset of $F(p)\cap P(A)$ need not be open, and hence $F(p)$ is not isomorphic to the quasi-state space of a C^*-algebra. Explicitly, any element h of $SQC(p)$ (or $QC(p)$) must have φ definite on h_∞.

Lemma 7.7. *If A is a C^*-algebra, p is a projection in A^{**}, and p satisfies (G), then $\pi^{**}(pAp)\subset\mathcal{K}(H_\pi)$ for every irreducible representation π of A.*

Proof. Part of the proof of 4.1 applies: If (e_n) is an orthonormal sequence in the range of $\pi^{**}(p)$ and $\psi_n = (\pi(\cdot)e_n, e_n)$, we can conclude that $\psi_n\to 0$. If $E_{[\epsilon,\infty)}(\pi^{**}(pap))$ has infinite rank for some a in A_+ and $\epsilon>0$, then we can obtain a contradiction by taking the e_n's in the range of $E_{[\epsilon,\infty)}(\pi^{**}(pap))$.

Corollary 7.8. *If A is a separable C^*-algebra, p is a projection in A^{**} satisfying the barycenter formula (in particular if p is closed), and p satisfies (G), then p is type I.*

Proof. Apply 5.13.

7.9. Continuation of Example 5.12.

(a) A subprojection p' of p is closed if and only if p' has finite rank or $p'\geq p_0$.

Proof. If p' has finite rank, then p' is closed by [1]. If $p'\geq p_0$, then $p'-p_0$ is closed in B^{**} by 0.1(ii). Therefore p' is closed in $\widetilde{A}^{**}$ and *a fortiori* in A^{**}.

If p' has infinite rank, then the range of p' contains an infinite dimensional subspace H' of $\overline{\text{span}}\{w_{n_1}, w_{n_2},\dots\} =$ range$(p-p_0)$. (This last is a codimension 1 subspace of $H_0 =$ range p.) Let (u_n) be a sequence of unit vectors in H' such that $u_n\xrightarrow{w} 0$ and $\psi_n = (\pi(\cdot)u_n, u_n)$. Then $\psi_n|_B\to 0$, since $\pi^{**}(pBp)\subset\mathcal{K}(H_\pi)$. Since p is compact, it follows that $\psi_n\to\varphi_0$. If p' is closed, this implies $\varphi_0\in F(p')$ and hence $p'\geq p_0$.

(b) For any non-zero closed subprojection p' of p there is a minimal projection p_1 such that $p_1\leq p'$ and $p'-p_1$ is closed.

Proof. If p' has infinite rank, then $p'\geq p_0$. We can find a minimal projection p_1 such that $p_1\leq p'-p_0$. Then $p_0\leq p'-p_1$ so that $p'-p_1$ is closed. If p' has finite rank then $p'-p_1$ is closed for any choice of p_1.

(c) If $pA^{**}p$ is identified with $B(H_0)$, then

$$\begin{aligned} pAp &= \{x\in B(H_0) : x-\varphi_0(x)I_{H_0}\in\mathcal{K}(H_0)\}\\ &= \{x\in B(H_0) : x-(xv_0, v_0)I_{H_0}\in\mathcal{K}(H_0)\}.\end{aligned}$$

Proof. Let $H_1 = H_0\ominus\mathbb{C}v_0$. By construction and the proof of 7.2, applied to B and $p-p_0$, $pBp=\mathcal{K}(H_1)$. Note that since p is compact, $pAp = p\widetilde{A}p$. To show that pAp is contained in the set indicated, it is enough to show pap is compact when

$a \in \widetilde{A}$ and $\varphi_0(a) = 0$. By [25, 3.13.6], $a = l + r$, where $l \in \widetilde{A}B$ and $r \in B\widetilde{A}$. Since x is compact if and only if x^*x is compact, $Bp \subset \mathcal{K}(H_\pi)$; and similarly $pB \subset \mathcal{K}(H_\pi)$. Therefore $pap \in \mathcal{K}(H_0)$.

For the reverse inclusion, since $p \in pAp$, $\mathcal{K}(H_1) \subset pAp$, and pAp is self-adjoint, it is sufficient to show that pAp contains every rank 1 operator x of the form $v \to (v, v_0)v_1$ for $v_1 \in H_1$. By the Kadison transitivity theorem ([20]) there is a in A such that $av_0 = v_1$ and $a^*v_0 = 0$. By the above, since $(av_0, v_0) = 0$, pap is compact. Hence $(pap - x) \in \mathcal{K}(H_1) = pBp$. This implies that x is in $p\widetilde{A}p$.

Since by [7, 4.4], the bidual of pAp is $pA^{**}p$, and since the predual of a W^*-algebra is unique, it follows from (c) that the dual space of pAp is $\mathcal{T}(H_0)$, the set of trace class operators on H_0. A concrete statement of this reads:

If v_0 is a unit vector in the (separable, infinite dimensional) Hilbert space H_0 and

$$A_{v_0} = \{x \in B(H_0) : x - (xv_0, v_0)I_{H_0} \text{ is compact}\},$$

then the dual space of A_{v_0} is naturally isometrically isomorphic to $\mathcal{T}(H_0)$. In particular, for $T \in \mathcal{T}(H_0)$, $\|T\|_1 = \sup\{|Tr(Tx)| : x \in A_{v_0}, \|x\| \leq 1\}$.

It is amusing to give a direct proof of this statement (which removes the parenthetical part of the hypothesis). The main step is to prove the second sentence.

Finally, we note that 5.12 gives another example of how the behavior of closed faces of C^*-algebras differs from that of C^*-algebras. If π is an irreducible representation of a C^*-algebra A, then $\pi(A) \cap \mathcal{K}(H_\pi)$ is either 0 or $\mathcal{K}(H_\pi)$. The analogous statement for Example 5.12 (replacing A by pAp) is false.

References

1. C. A. Akemann, *The general Stone-Weierstrass problem*, J. Funct. Anal. **4** (1969), 277–294.
2. ———, *Left ideal structure of C^*-algebras*, J. Funct. Anal. **6** (1970), 305–317.
3. ———, *Approximate units and maximal abelian C^*-subalgebras*, Pac. J. Math. **33** (1970), 543–550.
4. ———, *A Gelfand representation theory for C^*-algebras*, Pac. J. Math. **39** (1971), 1–11.
5. C. A. Akemann, J. Anderson, and G. K. Pedersen, *Approaching infinity in C^*-algebras*, J. Operator Theory **21** (1989), 255–271.
6. ———, *Excising states of C^*-algebras*, Canad. J. Math. **38** (1986), 1239–1260.
7. C. A. Akemann, G. K. Pedersen,and J. Tomiyama, *Multipliers of C^*-algebras*, J. Funct. Anal. **13** (1973), 277–301.
8. C. A. Akemann and F. Shultz, *Perfect C^*-algebras*, Memoirs A.M.S. **326** (1985).
9. A. S. Amitsur and J. Levitzki, *Minimal identities for algebras*, Proc. Amer. Math. Soc. **1** (1950), 449–463.
10. C. J. K. Batty, *Abelian faces of state spaces of C^*-algebras*, Commun. Math. Phys. **75** (1980), 43–50.
11. L. G. Brown, *Semicontinuity and multipliers of C^*-algebras*, Canad. J. Math. **40** (1988), 865–988.
12. ———, *Determination of A from $M(A)$ and related matters*, C.R. Math. Rep. Acad. Sci. Canada **10** (1988), 273–278.
13. ———, *Nearly relatively compact projections in operator algebras*, unpublished manuscript.
14. ———, *Semicontinuity and closed faces of C^*-algebras*, in preparation.
15. E. G. Effros, *Order ideals in a C^*-algebra and its dual*, Duke Math. J. **30** (1963), 391–412.
16. J. Glimm, *A Stone-weierstrass theorem for C^*-algebras*, Ann. Math. **72** (1960), 216–244.
17. ———, *Type I C^*-algebras*, Ann. Math. **73** (1961), 572–612.
18. H. E. Jensen, *Scattered C^*-algebras*, Math. Scand. **41** (1977), 308–314.
19. ———, *Scattered C^*-algebras II*, Math. Scand. **43** (1978), 308–310.
20. R. V. Kadison, *Irreducible operator algebras*, Proc. Nat. Acad. Sci. USA **43** (1957), 273–276.
21. I. Kaplansky, *Groups with representations of bounded degree*, Canad. J. Math. **1** (1949), 105–112.
22. E. Kirchberg, *Unitaries in UHF algebras and functorial properties of exactness*, J. reine angew. Math. **452** (1994), 39–77.
23. G. W. Mackey, *The Theory of Unitary Group Representations*, University of Chicago Press, Chicago and London, 1976.
24. ———, *Borel structures in groups and their duals*, Trans. Amer. Math. Soc. **85** (1957), 134–165.
25. G. K. Pedersen, *C^*-algebras and their Automorphism Groups*, Academic Press, London, 1979.
26. ———, *Applications of weak* semicontinuity in C^*-algebra theory*, Duke Math. J. **39** (1972), 431–450.
27. I. Raeburn and D. P. Williams, *Morita Equivalence and Continuous Trace C^*-algebras*, Mathematical Surveys and Monographs, vol. 60, American Mathematical Society, Providence R.I., 1998.

Dept. of Mathematics,
Purdue University
W. Lafayette, IN 47907, USA
e-mail: lgb@math.purdue.edu.